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Mechanics of collisional motion of granular materials. Part 1. General hydrodynamic equations

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Collisional motion of a granular material composed of rough inelastic spheres is analysed on the basis of the kinetic Boltzmann–Enskog equation. The Chapman–Enskog method for gas kinetic theory is modified to derive the Euler-like hydrodynamic equations for a system of moving spheres, possessing constant roughness and inelasticity. The solution is obtained by employing a general isotropic expression for the singlet distribution function, dependent upon the spatial gradients of averaged hydrodynamic properties. This solution form is shown to be appropriate for description of rapid shearless motions of granular materials, in particular vibrofluidized regimes induced by external vibrations.

The existence of the hydrodynamic state of evolution of a granular medium, where the Euler-like equations are valid, is delineated in terms of the particle roughness, β, and restitution, e, coefficients. For perfectly elastic spheres this state is shown to exist for all values of particle roughness, i.e. $-1 \leqslant \beta \leqslant 1$. However, for inelastically colliding granules the hydrodynamic state exists only when the particle restitution coefficient exceeds a certain value $e_m(\beta) < 1$.

In contrast with the previous results obtained by approximate moment methods, the partition of the random-motion kinetic energy of inelastic rough particles between rotational and translational modes is shown to be strongly affected by the particle restitution coefficient. The effect of increasing inelasticity of particle collisions is to redistribute the kinetic energy of their random motion in favour of the rotational mode. This is shown to significantly affect the energy partition law, with respect to the one prevailing in a gas composed of perfectly elastic spheres of arbitrary roughness. In particular, the translational specific heat of a gas composed of inelastically colliding ($e = 0.6$) granules differs from its value for elastic particles by as much as 55%.

It is shown that the hydrodynamic Euler-like equation, describing the transport and evolution of the kinetic energy of particle random motion, contains energy sink terms of two types (both, however, stemming from the non-conservative nature of particle collisions): (i) the term describing energy losses in incompressibly flowing gas; (ii) the terms accounting for kinetic energy loss (or gain) associated with the work of pressure forces, leading to gas compression (or expansion). The approximate moment methods are shown to yield the Euler-like energy equation with an incorrect energy sink term of type (ii), associated with the 'dense gas effect'. Another sink term of the same type, but associated with the energy relaxation process occurring within compressed granular gases, was overlooked in all previous studies.

The speed of sound waves propagating in a granular gas is analysed in the limits of low and high granular gas densities. It is shown that the particle collisional properties strongly affect the speed of sound in dense granular media. This dependence is manifested via the kinetic energy sink terms arising from gas compression. Omission

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of the latter terms in the evaluation of the speed of sound results in an error, which in the dense granular gas limit is shown to amount to a several-fold factor.

1. Introduction

Flows of granular materials are widely met in Nature and in various industrial technological processes. Examples are provided by snow and rock avalanches, and transport, mixing and screening of bulk materials (e.g. grain, coal, ore, etc.). Motion of granular media may occur in several regimes, which can be subdivided into rapid and slow flows. The latter flows are characterized by permanent contacts between the particles during their motion. In this regime bulk properties of moving granular media are controlled by the Coulomb interparticle friction forces. On the other hand, in rapid flows particles interact by fast impacts occurring during their collisions; most of the time particles freely fly between successive collisions. Transfer of particle kinetic energy and momentum within a rapidly flowing granular medium occurs during these collisions, the nature of which governs the effective medium transport properties.

The behaviour of rapidly flowing granular materials is similar to that of flowing liquids or gases, yet there exists a disparity between the motions of bulk materials and of a fluid continuum. This disparity pertains to the fundamental difference between the interparticle interactions within the former, and molecular interactions within the latter medium. Molecules composing liquids and gases interact with each other in a conservative manner, without losing their mechanical energy. On the other hand, particle impacts within granular media are accompanied by kinetic energy losses, associated with inelasticity of collisions and surface roughness. The effect of these losses is to increase the particle internal energy, and, hence, their temperature. Therefore, a constant source of mechanical energy is needed to sustain the collisional regime of a moving granular material.

Rapid granular flows may be distinguished with respect to the nature of external energy sources supplying kinetic energy to the moving particles. These sources include (i) gravity force, which, in particular, causes rapid shear flows of granular materials on inclined surfaces; (ii) air pressure, which governs particle motion in the processes occurring during pneumotransport, in fluidized beds, etc.; (iii) externally applied electric or magnetic fields (e.g. in electro- and magnetofluidized beds); (iv) externally induced vibrations (e.g. in vibrocrashing, vibroseparation and vibrofluidization processes).

Collisional motion of granular media may be investigated by stochastic analyses of an ensemble of identical (in most cases spherical or disk-like) particles, possessing specified inelasticity and roughness. These particle properties appear in various collisional models employed to describe motion of granular materials by the hydrodynamic equations. Effective transport properties required in these equations are obtained by various averaging methods.

Specific studies dealing with stochastic analyses of moving granular materials have been concerned with modelling of fluidized (Goldshtik & Kozlov 1973; Nigmatullin 1978), magnetofluidized (Buevich, Sutkin & Tetukhin 1984), and vibrofluidized beds (Raskin 1975), and rapid granular flows (Campbell 1990). The latter problem has received wide attention in the literature and has been treated by various methods of different rigour and complexity. In most studies the flow was assumed one-dimensional and particle–air interactions were neglected. The various methods tested on this model problem were also applied to modelling other important types of collisional granular motion, including fluidized beds (Homsy, Jackson & Grace 1992).

The methods used in studying rapid granular flows include (i) physical or experimental modelling (Drake 1990), (ii) computer simulations (Campbell & Brennen 1985; Campbell 1989), and (iii) gas kinetic theory (Jenkins & Savage 1983; Lun *et al.* 1984; Jenkins & Richman 1985*a*, *b*; Lun & Savage 1986, 1987; Richman 1989; Lun 1991). It must be noted, however, that complicated experimental measurements and computer simulations may be performed only in simple cases of granular flows. In this paper the kinetic theory is applied to modelling the collisional regime of moving granular materials.

Application of the kinetic theory methods to describing the collisional motion of granules include elementary theories (Haff 1983) and moment methods (e.g. Lun *et al.* 1984; Jenkins & Richman 1985*a*). Elementary kinetic theories, even in the classical cases of simple dilute gases, yield only qualitative results since their accuracy cannot be evaluated within the framework of the given method. Additional difficulties arise during applications of the elementary kinetic theories to dense gases, where dimensionality principles cannot be used for determination of the effective granular gas properties (Haff 1983).

The moment methods are based on transport equations obtained from the Boltzmann equation by integrating it with various weight functions (Grad 1949). In addition, these methods require *ad hoc* approximations of the singlet distribution function, f, appearing in the Boltzmann equation. Success or failure of these methods, thus, essentially depends upon the choice of the form of the singlet distribution function. Approximations of f used in various studies of granular materials include (i) the Grad approximation in terms of Hermittian polynomials (Jenkins & Richman 1985*a*, *b*) and (ii) the Sonine polynomials approximation (Lun 1991), based on known solutions for simple gas systems (Chapman & Cowling 1970). In both of the above approximations the Maxwell–Boltzmann distribution serves as the leading-order term.

In most fast shear flows, these approximations of the singlet distribution function cannot yield a satisfactory description of the effective properties governing average transport of moving granular materials because the latter approximation of f leads to a stress tensor, possessing equal normal components, which result disagrees with experimental measurements and computer simulations (Campbell 1990). In this respect we mention the work of Richman (1989), who proposed a more complicated approximation of f.

Moreover, theoretical solutions, based on the singlet distribution function expressed as Hermittian or Sonine polynomial approximations, may be used for description of flows characterized by moderate Mach numbers ($M < 1.851$) and small Knudsen numbers ($Kn \ll 1$) (Cercignani 1975). This clearly disagrees with the experimental and computational results obtained for fast granular flows (e.g. Campbell 1989), which show that the shear stress and the granular temperature T prevailing therein are of order of the square of the shear rate, i.e. $T \sim O[\sigma(u_0/L)]^2$, where σ is the particle radius and u_0/L is the characteristic value of the velocity gradient. Using the well-known estimates $a \sim T^{1/2}$, $\lambda \sim \sigma$ for the speed of sound, a, and the mean free path, λ, within the agitated dense granular media, one can rewrite the latter result in dimensionless form as $MKn = O(1)$.

An example of flows which do satisfy the above requirements with respect to the Mach the Knudsen numbers is rapid collisional motions induced by external vibrations. For these flows the shear pressure is found to be proportional to the share rate (Chlenov & Mikhailov 1972; Savage 1988), rather than to the square of the share rate.

Applications of vibration to the transport and handling of various bulk materials has received wide attention in numerous studies (Chlenov & Mikhailov 1972; Gutman 1968).

Can any rapidly moving granular system be described by hydrodynamic equations written in terms of pertinent effective transport properties? For molecular gases, such a description is valid for sufficiently long times, when inhomogeneities introduced by external sources eventually smooth out and the so-called local thermodynamic equilibrium prevails. While for molecular gases such a timescale is normally very short, this may not be the case in systems composed of coarse granules. For granular systems this equilibration process is accompanied by kinetic energy dissipation during particle collisions. If the latter dissipation process occurs sufficiently fast, the hydrodynamic state may be never reached, i.e. the evolution of such granular systems cannot be described by hydrodynamic equations.

A rigorous theory of the collisional motion of granular materials should determine the range of parameters in which the hydrodynamic equations adequately describe coarse-scale transport processes in granular media. The critical stage in the derivation of the hydrodynamic equations of granular motion is obtaining a true form of the singlet distribution function, which fully determines the structure of these equations, as well as the transport properties appearing therein. Since the moment methods hinge upon *a priori* assuming the functional form of f, these methods are incapable of substantiating the structure of the hydrodynamic equations, or allowing scrutiny of their validity range. These shortcomings are inherent to all the moment methods and limit their predictive capacity and range of applicability, which should be checked either by comparison with experimental data or with the results of computer simulations.

The objectives of this study are (i) to delineate the range of particle collisional properties, i.e. inelasticity and roughness, where the hydrodynamic state of the system evolution exists, and (ii) to derive the Euler-like hydrodynamic equations for the collisional motion of granular materials, using a rigorous mathematical method of the Chapman–Enskog type, based on a systematic solution of the Boltzmann equation; (iii) to verify the validity of the corresponding equations obtained by approximate (moment) kinetic methods.

One possible application of the Euler-like equations of granular motion is to flows induced by external vibrations (Goldshtein *et al.* 1993, 1994*a*). This type of flow is characterized by the appearance of shock waves, which govern the mechanism dominating kinetic energy transformation in flowing granular materials. Application of the Euler-like equations of granular motion to the problem of the vibrofluidized motion of layers composed of inelastic rough granules is a subject of the forthcoming papers of this series (Goldstein *et al.* 1994*a*, *b*).

2. Kinetic equation of Boltzmann–Enskog type

2.1. *Collisional model and collisional integral*

Consider an ensemble of identical rough spherical particles – granules of diameter σ with spherically symmetric mass distribution, performing chaotic translational and rotational motions in an effectively infinite spatial domain. The particles are assumed to be sufficiently heavy that the effect of the drag force (resulting from interactions with the surrounding gas) on their motion is negligible. The inertial properties of each particle are characterized by its mass m, moment of rotary inertia I, or dimensionless moment of inertia $k = 4I/m\sigma^2$. In particular, for uniform spheres $I = 0.1m\sigma^2$ and

$k = 0.4$. The dynamic state of any of the above particles is fully described by the location of its centre of mass $\boldsymbol{x}$, and by its linear, $m\boldsymbol{v}$ and angular, $I\boldsymbol{\omega}$ momenta. In the above $\boldsymbol{v}$, $\boldsymbol{\omega}$ are the particle translational and rotational velocities, respectively.

To complete the definition of the particle ensemble, changes of the dynamic variables resulting from particle interactions should be described. Particles are assumed to interact with each other only at contact, i.e. during collisions. We will utilize the hypothesis of stereomechanic impact (Goldsmith 1960; Lun & Savage 1987), which implies that particle impacts occur instantaneously and in such a way that the particle relative velocity $\boldsymbol{g}'_{21}$ after each collision depends only on their relative velocity, $\boldsymbol{g}_{21}$ just prior to the collision. Lun & Savage (1987) employed an additional assumption that vectors $\boldsymbol{g}_{21}$ and $\boldsymbol{g}'_{21}$ lie in the same plane, i.e.

$$(\boldsymbol{k}\cdot\boldsymbol{g}'_{21}) = -e(\boldsymbol{k}\cdot\boldsymbol{g}_{21}), \quad (\boldsymbol{k}\times\boldsymbol{g}'_{21}) = -\beta(\boldsymbol{k}\times\boldsymbol{g}_{21}). \qquad (1\,a, b)$$

Here, $\boldsymbol{k}$ is the unit vector directed from the centre of particle 2 to the centre of particle 1 at the moment of collision. Physical considerations (Johnson 1982), and experimental data (Goldsmith 1960) show that the normal restitution, or inelasticity coefficient, e depends upon the normal component of the impact velocity $g_{21k} = \boldsymbol{g}_{21}\cdot\boldsymbol{k}$ (or $v_{21k} = \boldsymbol{v}_{21}\cdot\boldsymbol{k}$; see (A 1) in Appendix A), where $\boldsymbol{v}_{21} \equiv \boldsymbol{v}_2 - \boldsymbol{v}_1$, and upon the properties of the particle material. According to the definitions of properties e, β, given in Appendix A, $-1 < \beta < 1$ and $0 < e < 1$.

It should be noted that the above collisional model provides but a simplified description of particle impacts. The roughness coefficient, β, generally depends upon the surface friction coefficient, as well as upon the normal, g_{21k}, and the tangential, $g_{21\tau}$, components of the relative impact velocity (Maw, Barber & Fawcett 1981; Sondergaard, Chaney & Brennen 1990); i.e. generally, $e = e(v_{21k})$, $\beta = \beta(v_{21k}, g_{21\tau})$. The interparticle surface friction coefficient is, however, difficult to measure (Sondergaard *et al.* 1990) and to incorporate in the collision integral (Campbell 1989). As an alternative, Jenkins (1992) accounted for elastic deformations associated both with normal and tangential displacements of the contact area, thereby employing two coefficients of restitution, associated with the corresponding velocity components. However, mathematically the tangential coefficient of restitution, introduced by Jenkins (1992) coincides with β employed in the present model, albeit with restriction $0 < \beta < 1$.

Equations ($1\,a, b$) were employed in classical kinetic theories of dense gases and liquids in two limiting cases: (i) $e = 1$, $\beta = -1$ and (ii) $e = 1$, $\beta = 1$, respectively describing perfectly smooth and rough (Chapman & Cowling 1970) spheres. In these cases the total kinetic energy of the system is conserved. However, for all other values of the coefficients β, e the collisional model ($1\,a, b$) predicts dissipation of the granular kinetic energy.

Collisional models employed in most studies of granular motion are characterized by constant parameters e, β. Raskin (1975), Jenkins & Savage (1983), Lun *et al.* (1984), Jenkins & Richman (1985*a, b*) examined the case of perfectly smooth inelastic spheres ($\beta = -1, 0 < e < 1$), where the tangential velocity components of the impacting particles do not change during the collision. The case $\beta = 0$, corresponding to the situation where maximum kinetic energy is dissipated, has been analysed by computer simulations (Campbell 1989; Campbell & Brennen 1985), which found surface friction and inelasticity to be sufficiently large to eliminate the postcollisional tangential relative velocities. Goldshtik & Kozlov (1973) and Nigmatullin (1978) used a model of perfectly rough ($\beta = 1$), inelastic ($e < 1$) spheres for describing fluidized bed processes. The collisional model ($1\,a, b$) with arbitrary constant parameters e, β was used by

Jenkins & Richman (1985*b*) for disk-like particles, and by Lun & Savage (1987), Goldshtein, Poturaev & Shulyak (1990), and Lun (1991) for spheres.

Lun & Savage (1986) treated the case of smooth spheres with the restitution coefficient exponentially dependent on the normal component of the relative particle impact velocity. In this section we will consider the collisional model (1 *a*, *b*), in which both parameters e, β arbitrarily depend on the particle relative collisional speed.

The formalism considered here is subjected to the requirement of the mathematical simplicity of the relationship $g'_{21} = g'_{21}(g_{21})$ between the relative velocities before and after collisions. This means that an inverse function, $g_{21} = g_{21}(g'_{21})$ exists. In addition, the following assumptions will be used:

(i) Statistical expectation values of the particle ensemble may be expressed via the singlet distribution function $f_1 = f(x, v_1, \omega_1, t)$; pair correlations of each two particles may be expressed via the pair distribution function $f^{(2)}(x_1, \tau_1; x_2, \tau_2; t)$, where $\tau_i = (v_i, \omega_i)$, $i = 1, 2$.

(ii) The probabilities of triple and multiple collisions are negligible.

(iii) No external forces act upon the particles.

It will be shown below that the treatment to be proposed may be generalized to include the gravity force, the effect of which is to add the free-fall movement to the whole particle ensemble. Non-trivial effects of the gravity force are related to interactions between the ensemble (e.g. granular layer) and a solid wall (Goldshtein *et al.* 1994*a*).

Based on the above assumptions and using the technique employed in the classical kinetic theory (Chapman & Cowling 1970; Condiff, Lu & Dahler 1965), one obtains the equation of change for the singlet distribution function f_1 (see Appendix C†):

$$\frac{\partial f_1}{\partial t} + v_1 \cdot \frac{\partial f_1}{\partial x} = \frac{\partial_c f}{\partial t}, \tag{2}$$

where the collisional term is given by

$$\frac{\partial_c f}{\partial t} = \int \mathrm{d}^3 v_2\, \mathrm{d}^3 \omega_2\, \mathrm{d}^2 k S(k \cdot v_{21}) [\Lambda f^{(2)}(x, \tau''_1; x + \sigma k, \tau''_2; t) - f^{(2)}(x, \tau_1; x - \sigma k, \tau_2; t)], \tag{3}$$

$$\Lambda = -\partial(v''_1, v''_2, \omega''_1, \omega''_2)/\partial(v_1, v_2, \omega_1, \omega_2)\, \tilde{e}^{-1}. \tag{4}$$

In the above, the coefficient $\tilde{e} = \tilde{e}(v_{21k})$ and the double-primed dynamic variables are defined in (A 9), (A 10) in Appendix A, t is the time variable, and $S(k \cdot v_{21}) = \sigma^2(k \cdot v_{21})\, \theta(k \cdot v_{21})$ (θ is the Heaviside function, i.e. $\theta(x) = 0$ for $x < 0$, $\theta(x) = 1$ for $x > 0$).

Equations (2)–(4) are not closed, sine they include f and $f^{(2)}$. We will obtain a closing relation for the singlet distribution function (Enskog equation) by invoking the generally accepted assumption of molecular chaos (see for example McCoy, Sandler & Dahle 1966), modified to include the Enskog's frequency factor $g(n)$:

$$f^{(2)}[x + \sigma k(1-\lambda), \tau_1; x - \lambda\sigma k, \tau_2; t] = f_1[x + \sigma k(1-\lambda)] f_2(x - \lambda\sigma k)\, g[x + \sigma k(\tfrac{1}{2} - \lambda)], \tag{5}$$

where the dependence of g upon its argument, appearing in (5), is implicit, i.e. $g = g\{n[x + \sigma k(\frac{1}{2} - \lambda)]\}$.

† The following appendices are available from authors or the JFM editorial office upon request: Appendix C. Conservation Laws. Appendix D. Existence and uniqueness of the hydrodynamic solution. Appendix E. Estimation of the Maxwell–Boltzmann approximation.

In the above, $f_i(x) = f(x, \tau_i, t)$, λ is an arbitrary parameter, and Enskog's frequency function $g(n)$ is the equilibrium radial distribution function at contact (Chapman & Cowling 1970). This function is independent of particle collisional properties and may be calculated, for example, from approximate formulae of Carnahan & Starling (1969).

Equations (2)–(5) constitute Enskog's theory generalized here to the case of inelastically colliding rough particles, where the kinetic energy is not conserved owing to dissipation. These equations will serve here as *modus operandi* for derivation of the equations governing hydrodynamic properties of the particle ensemble.

Enskog's model is expected to adequately describe the behaviour of systems containing a large number of macroscopic particles in cases where the attractive part of their interaction potential (e.g. adhesion forces) may be neglected. This assertion is supported by the success of Enskog's theory in describing properties of simple dense fluids. In particular, viscosity and thermal conductivity calculated by this theory differ from the corresponding quantities measured for hydrogen and argon by less than 15% (Hanlay, McCarthy & Cohen 1972). A close agreement between the measured properties of those calculated with a corresponding choice of the function g, is also found for gas–liquid phase transitions.

2.2. *Laws of conservation*

Multiply both sides of (2) by an arbitrary intensive property $\psi = \psi(\tau_1)$ and integrate over the whole range of the variable $\tau_1 = (v_1, \omega_1)$. Then, after performing some algebraic manipulations (see Appendix C) one obtains the general equation of change

$$\frac{\partial}{\partial t}\langle\psi\rangle = -\frac{\partial}{\partial x}\cdot J(\psi) + I(\psi), \tag{6a}$$

with the ensemble average defined by

$$\langle\psi\rangle = \int d^6\tau f(x, \tau, t)\, \psi(\tau), \tag{6b}$$

and where $I(\psi)$ and $J(\psi)$ can be identified as the volumetric source term and the flux vector of the macroscopic variable $\langle\psi\rangle$, respectively. In general, the decomposition

$$J(\psi) = J^{(k)}(\psi) + J^{(c)}(\psi)$$

is valid, i.e. the flux of the property ψ consists of a kinetic or diffusional part

$$J^{(k)}(\psi) \equiv \langle v_1 \psi\rangle, \tag{7}$$

and a collisional transfer part of the form

$$J^{(c)}(\psi) \equiv \frac{\sigma}{4}\int d^6\tau_1\, d^6\tau_2\, d^2k \int_0^1 d\lambda S(k\cdot v_{21})\, k\, \Delta'\psi f^{(2)}(x+\sigma k(1-\lambda), \tau_1; x-\lambda\sigma k, \tau_2; t). \tag{8}$$

Here $\Delta'\psi = \Delta\psi_1 - \Delta\psi_2 = (\psi_1' - \psi_1) - (\psi_2' - \psi_2)$ is the change of the property ψ associated with 'direct collisions' (see Appendix A for the definition of these events). Finally, the source term is given by

$$I(\psi) \equiv \frac{1}{2}\int d^6\tau_1\, d^6\tau_1\, d^6\tau_2\, d^2k\, S(k\cdot v_{21})\, \Delta\psi f^{(2)}(x+\sigma k, \tau_1; x, \tau_2; t), \tag{9}$$

where $\Delta\psi = (\psi_1' - \psi_1) + (\psi_2' - \psi_2)$.

Formula (9) describes the collisional rate of change of $\langle\psi\rangle$. For the model investigated here, this expression takes into account transfer of particle kinetic energy between their rotational and translational degrees of freedom, and also the total energy losses (see (A 6)).

Employing (6) with ψ being particle mass, m, linear momentum, $m\boldsymbol{v}$, and kinetic energy of random motion $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$, one obtains the following conservation equations governing the evolution of these properties:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}} \cdot (\rho \boldsymbol{u}) = 0, \quad \frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_i} t_{ij}(\boldsymbol{x}, t) = 0, \tag{10a, b}$$

$$\frac{\partial}{\partial t}[ne_0 + \tfrac{1}{2}\rho u^2] + \frac{\partial}{\partial \boldsymbol{x}} \cdot \boldsymbol{q}(\boldsymbol{x}, f) = I(E). \tag{10c}$$

In these equations $\rho = mn$ is the bulk mass density, $\boldsymbol{u} = \langle \boldsymbol{v} \rangle / n$ is the bulk velocity (the mean particle spin velocity $\boldsymbol{\omega}_0 \equiv \langle \boldsymbol{\omega} \rangle / n = 0$) and $ne_0 = \langle \frac{1}{2}m|\boldsymbol{v}-\boldsymbol{u}|^2 + \frac{1}{2}I\omega^2 \rangle = n(e_{0t} + e_{0r})$ is the particle average fluctuation kinetic energy. The momentum flux t_{ij} consists of macroscopic, $\rho u_i u_j$, and microscopic (pressure tensor, P_{ij}) parts:

$$t_{ij}(\boldsymbol{x}, f) = \rho u_i u_j + P_{ij}(\boldsymbol{x}, f). \tag{10d}$$

The expression for the kinetic energy flux, q_i, is

$$q_i(\boldsymbol{x}, t) = nu_i(e_0 + \tfrac{1}{2}mu^2) + u_j P_{ij}(\boldsymbol{x}, f) + j_i(\boldsymbol{x}, f), \tag{10e}$$

in which the second term accounts for work of pressure forces.

The kinetic parts of the pressure tensor, $P_{ij}^{(k)}$ and of the vector heat flux, $j_i^{(k)}$ do not depend upon the choice of the collisional model, employed in the derivation of (10*a–c*). Comparable collisional transfer parts, $P_{ij}^{(c)}, j_i^{(c)}$, expressed by integral (8) (with appropriate choice of ψ) depend upon the nature of the particle collisions embodied in $\Delta'\psi, \Delta\psi$. Using (5), $P_{ij}^{(c)}, j_i^{(c)}$ may be expressed via the singlet distribution function

$$\boldsymbol{P}^{(c)} = \langle\langle m\boldsymbol{v} \rangle\rangle, \quad \boldsymbol{j}^{(c)} = \langle\langle \tfrac{1}{2}m|\boldsymbol{v}-\boldsymbol{u}|^2 + \tfrac{1}{2}I\omega^2 \rangle\rangle. \tag{11a, b}$$

Here, for any function F the symbol $\langle\langle F \rangle\rangle$ indicates the integral

$$\tfrac{1}{4}\sigma \int \mathrm{d}^6\tau_1\, \mathrm{d}^6\tau_2\, \mathrm{d}^2k \int_0^1 \mathrm{d}\lambda\, S(\boldsymbol{k}\cdot\boldsymbol{v}_{21})\, k f_1[\boldsymbol{x} + \sigma\boldsymbol{k}(1-\lambda)] f_2(\boldsymbol{x} - \lambda\sigma\boldsymbol{k})\, g[\boldsymbol{x} + \sigma\boldsymbol{k}(\tfrac{1}{2}-\lambda)]\, \Delta' F,$$

and expressions for $\Delta'(m\boldsymbol{v})$, $\Delta'\frac{1}{2}(m|\boldsymbol{v}-\boldsymbol{u}|^2 + I\omega^2)$ are given by

$$\Delta'(m\boldsymbol{v}) = 2m\{\eta_2[\boldsymbol{v}_{21} - \tfrac{1}{2}\sigma\boldsymbol{k} \times (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)] + (\eta_1 - \eta_2)\,\boldsymbol{k}(\boldsymbol{v}_{21}\cdot\boldsymbol{k})\}, \tag{11c}$$

$$\begin{aligned}\Delta'(\tfrac{1}{2}mC^2 + \tfrac{1}{2}I\omega^2) = m\{(\eta_1 - \eta_2)\,[(\boldsymbol{k}\cdot\boldsymbol{C}_2)^2 - (\boldsymbol{k}\cdot\boldsymbol{C}_1)^2] + \eta_2[C_2^2 - C_1^2 \\ - \boldsymbol{k}\cdot(\boldsymbol{\omega}_2 \times \boldsymbol{C}_2 + \boldsymbol{\omega}_1 \times \boldsymbol{C}_1)]\} + m\eta_2 \tfrac{1}{4}\sigma^2[\omega_2^2 - \omega_1^2 - (\boldsymbol{k}\cdot\boldsymbol{\omega}_2)^2 + (\boldsymbol{k}\cdot\boldsymbol{\omega}_1)^2],\end{aligned} \tag{11d}$$

and $\boldsymbol{C}_i = \boldsymbol{v}_i - \boldsymbol{u}$, $i = 1, 2$.

Finally, the source term appearing in (10*c*) is given by

$$I(E) = I(f, f, \boldsymbol{x}) = \frac{1}{2}\int \mathrm{d}^6\tau_1\, \mathrm{d}^6\tau_2\, \mathrm{d}^2k S(\boldsymbol{k}\cdot\boldsymbol{v}_{21})\, g[n(\boldsymbol{x} + \tfrac{1}{2}\sigma\boldsymbol{k})] f_1(\boldsymbol{x} + \sigma\boldsymbol{k}) f_2(\boldsymbol{x})\, \Delta E, \tag{12}$$

with ΔE given by (A 6).

The conservation equations (10), written in terms of fluxes, possess divergent forms, which are convenient for formulating conditions for hydrodynamic quantities upon a shock wave (see §6). For describing continuous (shockless) flows the conservation equations are normally expressed via the microscopic parts of the fluxes, i.e. via the pressure tensor P_{ij} and the heat flux $\boldsymbol{j}$:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \boldsymbol{x}}\cdot(\rho \boldsymbol{u}), \quad \frac{\partial u_j}{\partial t} = -u_i \frac{\partial u_j}{\partial x_i} - \frac{1}{\rho}\frac{\partial P_{ij}}{\partial x_i}, \tag{13a, b}$$

$$\frac{\partial e_0}{\partial t} = -\boldsymbol{u}\cdot\frac{\partial e_0}{\partial \boldsymbol{x}} - \frac{P_{ij}}{n}\frac{\partial u_i}{\partial x_j} - \frac{1}{n}\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{j} + \frac{1}{n}I(f, f, \boldsymbol{x}). \tag{13c}$$

McCoy *et al.* (1966) first derived the conservation equations (13*a–c*) for granular media consisting of absolutely rough elastic spheres. Their expressions for the pressure tensor, heat flux vector and volume source term are obtainable from (10*d, e*) by a small-gradient expansion of the singlet distribution function f. Lun & Savage (1987) generalized the work of McCoy *et al.* (1966) to the case of arbitrary constants e, β. Goldshtein *et al.* (1990) obtained the conservation equations (13*a–c*) for the same collisional model, but for arbitrary gradients of f. Here, (13*a–c*) with expressions (10*d, e*) are further generalized for a more complicated collisional model (1).

In order to derive the hydrodynamic equations from (13*a–c*) it is necessary to determine the singlet distribution function and to calculate the appropriate constitutive equations for the pressure tensor, the heat flux and the source term. This will be done in §3 using a variant of the Chapman–Enskog method.

3. Hydrodynamics of the spatially homogeneous state of a system of rough inelastic spheres

3.1. *Solution scheme for the general collisional model*

The Chapman–Enskog method is based upon physical mechanisms governing evolution of a multiparticle system towards the equilibrium state. During this process the spatial and temporal dependences of the singlet distribution function f_1 smooth out (Bogoliubov 1948). In the absence of external forces and energy sinks (sources) the system subsequently passes the kinetic and the hydrodynamic stages. In the kinetic stage f_1 changes during a period comparable to the mean time t_0 between two successive particle collisions (mean free time); and the spatial inhomogeneity of f_1 is characterized by a scale comparable to the particle mean free path l.

In the hydrodynamic stage of the system evolution f_1 is characterized by the spatial, L, and temporal, t_1, length and time scales, where $L/t_1 \approx a$, the speed of sound in the gas. In the final equilibrium homogeneous state f_1 is a space- and time-independent function.

The above considerations should be revised before applying them to investigation of a system composed of inelastic rough spheres. Since in the latter case particle interactions are non-conservative and the particle thermodynamic temperature is not included in consideration, the equilibrium state of the system degenerates to the state where the kinetic energy of particle random motion is zero.

In a spatially homogeneous state, characterized by a non-zero fluctuation kinetic energy e_0, the latter quantity monotonically decreases in time. Therefore, even in the hydrodynamic stage of the system evolution (if such a stage exists) the fluctuation energy e_0, speed of sound a and the singlet distribution function should change during the time period of order t_0.

Bearing in mind the above, and in accordance with the basic idea of the Chapman–Enskog method (Chapman & Cowling 1970) we assume that after a certain characteristic time (which will be discussed later) there ensues a hydrodynamic stage of evolution of the system, in which stage its state is completely described by the hydrodynamic quantities

$$\zeta_\alpha(t) \equiv \{\xi_0(t), \xi_i(t), \xi_4(t)\} \equiv \{n(t), u_i(t), e_0(t)\}, \quad i = 1,2,3,$$

and the singlet distribution function of the system possesses the form

$$f(\boldsymbol{v}, \boldsymbol{\omega}, t) \underset{t \geqslant t_0}{=\!=} f(\boldsymbol{v}, \boldsymbol{\omega}, \xi_\alpha(t)), \quad \alpha = 0,1,2,3,4. \tag{14}$$

Obviously, for the spatially homogeneous state all functions do not depend on the spatial variable and, as follows from (13 *a*, *b*), the average values of the particle number density n and velocity $\boldsymbol{u}$ are constants. Consequently, we can express without loss of generality that the 'hydrodynamic' solution $f^{(0)}$ of the kinetic equations (2)–(5) possesses the form $f_1^{(0)} = f^{(0)}(n, \boldsymbol{C}_1, \boldsymbol{\omega}_1, e_0(t))$, wherein $\boldsymbol{C}_1 = \boldsymbol{v}_1 - \boldsymbol{u}$ denotes the peculiar velocity of translation (Chapman & Cowling 1970).

In the spatially homogeneous state the kinetic equation (2) and the collisional term (3), combined with the molecular chaos assumption (5), adopt the forms

$$\frac{\mathrm{d}f_1^{(0)}}{\mathrm{d}t} = J(f^{(0)}, f^{(0)}), \tag{15}$$

$$J(f^{(0)}, f^{(0)}) = g(n) \int \mathrm{d}^6\tau_2\, \mathrm{d}^2k S(\boldsymbol{k}\cdot\boldsymbol{v}_{21}) [\Lambda f^{(0)}(\boldsymbol{\tau}_1'', t) f^{(0)}(\boldsymbol{\tau}_2'', t) - f^{(0)}(\boldsymbol{\tau}_1, t) f^{(0)}(\boldsymbol{\tau}_2, t)]. \tag{16}$$

The solution $f_1^{(0)}$ of (15) depends upon several quantities which possess the dimensionality of velocity, namely $(e_0/m)^{1/2}$, and parameters $c_j (j = 1,2,\ldots,n)$, characterizing the dependence of the collisional properties e, β upon the relative particle impact velocity $\boldsymbol{g}_{21}$. Experimental correlations (Goldsmith 1960) show that e is a decreasing function of $v_{21k} = \boldsymbol{g}_{21}\cdot\boldsymbol{k}$. In particular, for small velocities ($v_{21k} < 0.06\ \mathrm{m\,s^{-1}}$), e may be approximated by $e \approx 1 - c_1 v_{21k}$ (Goldsmith 1960). With increasing velocity v_{21k} the slopes of the curves $e = e(v_{21k})$ decrease.

Imposing, so far, no limitations on the precise form of the functions $e = e(v_{21k})$, $\beta = \beta(v_{21k}, g_{21\tau})$, we will use dimensionality considerations to look for a solution in the form $f_1^{(0)} = n(mI)^{3/2} e_0^{-3} F(\boldsymbol{V}_1, \boldsymbol{\Omega}_1, \epsilon_j), j = 1,2,\ldots,n$, where the non-dimensional function F depends upon the scaled velocities $\boldsymbol{V}_1 = \boldsymbol{C}_1(m/e_0)^{1/2}$, $\boldsymbol{\Omega}_1 = \boldsymbol{\omega}_1(I/e_0)^{1/2}$ and parameters $\epsilon_j = c_j(m/e_0)^{1/2}$. Finally, assuming that $f^{(0)}$ is isotropic in the velocity space $\boldsymbol{\mu}_1 = (\boldsymbol{V}_1, \boldsymbol{\Omega}_1)$ (see a discussion of this matter in §3.2), one can find the singlet distribution function in the form

$$f_i^{(0)} = \frac{n(mI)^{3/2}}{e_0^3} F_i, \tag{17}$$

where $F_i = F(V_i^2, \Omega_i^2, \epsilon_1^2, \ldots, \epsilon_n^2)$, $i = 1,2$, and the time dependence of $f_i^{(0)}$ is implicitly embodied in $e_0 = e_0(t)$.

Introduce (17) into (13 *c*) to obtain the equation governing the evolution of the kinetic energy of particle random motion due to the dissipation process:

$$\frac{\mathrm{d}e_0}{\mathrm{d}t} = K(F,F)(lm^{1/2})^{-1} e_0^{3/2} \equiv I^{(0)}(f^{(0)})/n. \tag{18}$$

Here $l = (\sigma^2 g n)^{-1}$ is an analogue of the particle mean free path in dense gases, and

$$K(F,F) \equiv \frac{1}{2e_0}\int d^6\mu_1\, d^6\mu_2\, d^2k\theta(\boldsymbol{k}\cdot\boldsymbol{V}_{21})(\boldsymbol{k}\cdot\boldsymbol{V}_{21})\,F_1\,F_2\,\Delta E, \tag{19}$$

where $\boldsymbol{V}_{21} \equiv \boldsymbol{V}_2 - \boldsymbol{V}_1$ is the dimensionless relative translational impact velocity.

Introducing (17) for $f_i^{(0)}$, $i = 1,2$ in (15) and using collisional integral (16) and (18), one can find the equation for F in the form

$$-K(F,F)\left[3F_1 + V_1^2\frac{\partial F_1}{\partial V_1^2} + \Omega_1^2\frac{\partial F_1}{\partial \Omega_1^2} + \sum_{i=1}^{n}\epsilon_1^2\frac{\partial F}{\partial \epsilon_1^2}\right] = \tilde{J}(F,F), \tag{20}$$

where $\tilde{J}(F,F) = J^{(0)}(F,F)/[\sigma^2(m/e_0)^{1/2}(mI)^{3/2}e_0^{-6}g(n)]$

is the collisional integral (16) in non-dimensional form. Normalization conditions for F are

$$\int d^6\mu_1\,F_1 = \frac{1}{2}\int d^6\mu_1(V_1^2+\Omega_1^2)\,F_1 = 1, \tag{21}$$

which may be obtained from the definition (6*b*) of the basic hydrodynamic quantities via the distribution function and using the employed solution (17).

As a result of the self-similar solution (17), equation (15) adopts the form (20), which explicitly does not contain time t. The solution F of (20) depends upon the absolute values of velocities V_1, Ω_1 (rather than upon the vectors $\boldsymbol{V}_1$, $\boldsymbol{\Omega}_1$) and parameters ϵ_j. This solution satisfying the normalization conditions (21) together with the ϵ_j-dependent function $K(F,F)$ calculated from (19), may be substituted into (18). This will yield the dependence of the right-hand side of (18) upon e_0. As a result, (18) reduces to a form enabling calculation of the temporal evolution of e_0 for the selected particle collisional model. In the particular case e, $\beta = \text{const}$ one obtains that $K(F,F) = \text{const}$ (independent of e_0), in which case $e_0(t)$ will be obtained in the following section.

3.2. *Hydrodynamic stage of system evolution for smooth sphere model*

Equations (17)–(21) govern the hydrodynamic, spatially homogeneous state of a system composed of inelastic rough spheres. In this section we will investigate such a state of an agitated granular medium in the case $\beta = -1$, $e = \text{const}$. In this case particle rotational degrees of freedom degenerate and do not affect the hydrodynamic state. Therefore, equations governing the singlet distribution function for a smooth sphere system cannot be directly obtained from the comparable equations (17)–(21) by passing in the latter to the limit $\beta \rightarrow -1$. Therefore, we will start the treatment by modifying the collisional integral (16) appearing in (15). In this case the singlet distribution function $f^{(0)}$ does not depend upon particle angular velocity and collisional parameters c_i.

Combining solution (17), an appropriately simplified collisional integral (16), and noting that $\Lambda = 1/e^2$, one obtains

$$J(f^{(0)},f^{(0)}) = g(n)\int d^3v_2\, d^2kS(\boldsymbol{k}\cdot\boldsymbol{v}_{21})\left[\frac{1}{e^2}f^{(0)}(\boldsymbol{v}_1'',t)f^{(0)}(\boldsymbol{v}_2'',t) - f^{(0)}(\boldsymbol{v}_1,t)f^{(0)}(\boldsymbol{v}_2,t)\right], \tag{22}$$

$$f_i^{(0)} = \frac{n}{(e_0/m)^{3/2}}F_i, \quad F_i = F(V_i^2, e), \quad i = 1,2. \tag{23a, b}$$

Introducing (23a) into (13c) one obtains (18) with $K(F,F) = K_e(F,F)$ given by

$$K_e(F,F) = -\frac{\pi(1-e^2)}{16}\int d^3V_1\, d^3V_2\, F_1\, F_2\, V_{21}^3. \tag{24}$$

It follows from (23b) that F_i depends upon time only via V_i^2, and that $K_e(F,F)$ is time independent (depends only on parameter e). Therefore, the solution of the kinetic energy balance equation (18) with $K_e(F,F)$ given by (24) may be immediately calculated:

$$e_0(t) = \frac{e_{00}}{(1+t/\delta)^2}. \tag{25}$$

Here, $e_{00} = e_0(0)$, the timescale δ of energy decay is given by

$$\delta = -(2t_0)/K_e(F,F), \tag{26}$$

and $t_0 = (e_{00}/m)^{-1/2}l$ is the time comparable to the mean free time in the granular system. When the coefficient of restitution e is close to unity, $K_e(F,F) \ll 1$, and the above two characteristic times significantly differ.

One can see that the long-time decay of the particle random motion energy in the spatially homogeneous case is given by t^{-2}. This time dependence of e_0 was obtained by Raskin (1975) and Huff (1983), without, however, revealing the dependence of the dissipation process upon the restitution coefficient e. For this purpose it is necessary to evaluate $K_e(F,F)$ and, subsequently, the function F. The equation for calculating the latter function is derived by substituting (23a) into (15) with the collisional integral given by (22). Introducing (23a) for $f_i^{(0)}$ into (15) and using (22) together with (18), one obtains the equation for F (cf. (20))

$$-K_e(F,F)\left(\tfrac{3}{2}F_1 + V_1^2\frac{dF_1}{dV_1^2}\right) = \tilde{J}(F,F), \tag{27}$$

wherein $K_e(F,F)$ is given by (24) and

$$\tilde{J}(F,F) = \int d^3v_2\, d^2k\, \theta(\boldsymbol{k}\cdot\boldsymbol{V}_{21})(\boldsymbol{k}\cdot\boldsymbol{V}_{21})\left[\frac{1}{e^2}F_1''\, F_2'' - F_1\, F_2\right]. \tag{28}$$

Normalization condition (21) for the particular case of smooth particles ($\beta = -1$), considered now, adopts the form

$$\int d^3V_1\, F_1 = \frac{1}{2}\int d^3V_1\, V_1^2 F_1 = 1. \tag{29}$$

Equation (29) constitutes the necessary and sufficient condition for the solubility of the integro-differential equation (27) with $\tilde{J}(F,F)$ given by (28). A proof of the latter property together with the existence and uniqueness of the solution F is available from the authors upon request to interested readers.

For investigating the dependence of $K_e(F,F)$ upon the restitution parameter e, it is necessary to solve (27)–(29) for F, which will be done in the following section by expanding it in series of the Sonine polynomials, together with investigation of the series convergence (see also Condiff *et al.* 1965).

It must be finally noted that solution (17) was obtained by *a priori* assuming F_i to be an isotropic function. However, the latter assumption is, in fact, unnecessary since the isotropy of F_i and, hence, $f_i^{(0)}$ may be derived from the isotropy of the operator $J(f^{(0)}, f^{(0)})$ (see (22)) by using (23a) and (15) together with the arguments summarized in Appendix D.

3.3. *Solution for F by Sonine polynomials expansion*

Here we will evaluate the effect of restitution coefficient e on the singlet distribution function of a system of perfectly smooth spheres. Towards this goal expand function F_1 in a series in terms of Sonine polynomials

$$F_1 = F_1^{(0)} \sum_{i=0}^{\infty} a_i S_{1/2}^{(1)}(\tilde{V}_1^2), \quad \tilde{V}_1^2 = V_1^2/\alpha_t, \tag{30}$$

where a_i are dimensionless functions of parameter e, $F_1^{(0)}$ is the Maxwell–Boltzmann function

$$F_1^{(0)} = (\pi\alpha_t)^{-3/2} \exp(-V_1^2/\alpha_t),$$

and $S_m^{(n)}(x)$ are the Sonine polynomials

$$S_m^{(n)}(x) = (n!)^{-1} e^x \, \mathrm{d}^n(e^{-x} x^{m+n})/\mathrm{d}x^n,$$

normalized in such a way that

$$\int_0^\infty \mathrm{d}x \, x^m e^{-x} S_m^{(n)}(x) \, S_m^{(n')}(x) = \frac{\Gamma(m+n+1)\,\delta_{nn'}}{n!}. \tag{31}$$

Here $\alpha_t = 4/3$, $\Gamma(x)$ is the Gamma function and $\delta_{nn'}$ is the Kronecker delta. After introducing series (30) into condition (29) and using normalization conditions (31) imposed on $S_m^{(n)}(x)$, one obtains the first two coefficients of (30):

$$a_0 = 1, \quad a_1 = 0. \tag{32}$$

Other coefficients in expansion (30) may be found after employing the calculational scheme of the moment method (see e.g. Condiff *et al.* 1965), which will here be modified to the present nonlinear case. Towards this goal we will suppose, subject to *a posteriori* verification, that $a_i \ll 1$ ($i = 1, 2, 3, \ldots$) and, hence, $a_i a_j \ll a_j$ ($j = 1, 2, \ldots$). Limiting ourselves by one more approximation term in (30), following the Maxwell–Boltzmann term, we obtain

$$a_2 = \frac{16(1-e)(1-2e^2)}{64+(1-e)\,[(1+e^2)\,190+147]}. \tag{33}$$

Direct evaluation shows that $|a_2| \leqslant 0.04$ for all values of parameter e and, consequently, the above assumption of smallness of a_i is quantitatively substantiated. Details of calculations of (33) are available from the authors upon request for interested readers.

Substitution of expansion (30) with a_0, a_1 and a_2 given by (32) and (33) into (24) yields

$$K_e(F, F) = -\left(\frac{2\pi}{3}\right)^{1/2} \tfrac{4}{3}(1-e^2)\left(1+\frac{3a_2}{16}\right). \tag{34}$$

This approximation differs from the comparable lower-order approximation by the Maxwell–Boltzmann function (obtained from (30) with $a_2 = 0$) by less than 1 %. This means that the Euler-like equations governing the motion of a medium composed of inelastic smooth granules (Jenkins & Richman 1985*a*) are practically exact in the whole range of existence of the hydrodynamic solution.

The difference between the singlet distribution function $f^{(0)}$ and the Maxwell–Boltzmann formula produces only a weak effect on the hydrodynamic equation (18) of medium consisting of inelastic smooth granules. Bearing in mind the surprising accuracy of this lowest-order approximation of $f^{(0)}$ and the uniqueness of the hydrodynamic solution for $\epsilon \ll 1$ one can expect that such a solution for systems composed of rough inelastic granules exists and is unique also in a wider range of e. Limitations which should be imposed upon the applicability range of the hydrodynamic solution may be obtained by studying its stability.

3.4. *Rate of approach of a granular system to the hydrodynamic state and its stability*

Here we will investigate the temporal approach of any solution of the Boltzmann equation (15), (22) towards the hydrodynamic solution $f^{(0)}$ for the system of smooth inelastic spherical particles. For classical gaseous systems ($e = 1$) this question was studied for the linearized Boltzmann equation (see e.g. Cercignani 1975).

Let at $t = 0$ the above particle system be characterized by the distribution $f = f^{(0)}(1+h)$, slightly deviating by a small function h ($|h| \ll 1$) from the hydrodynamic homogeneous solution $f^{(0)}$ of (15). Then one can obtain the following equation governing the evolution of h:

$$\frac{\partial(f_1^{(0)} h_1)}{\partial t} = J(f^{(0)}, f^{(0)}h), \tag{35}$$

where $J(f^{(0)}, f^{(0)}h)$ is the linearized collision operator (22). This linear equation possesses a self-similar solution (cf. the hydrodynamic solution (23a) for $f^{(0)}$):

$$h(t, V_i, e) = \left(\frac{e_{00}}{e_0}\right)^{\gamma} H_i, \quad H_i = H(V_i, e), \quad i = 1, 2, \tag{36}$$

where γ is an e-dependent function to be determined below. Substituting (36) into (35) and using (25) for $e_0(t)$, one obtains the following eigenvalue problem:

$$-K_e(F,F)\left[(\tfrac{3}{2}+\gamma) F_1 H_1 + H_1 V_1^2 \frac{\partial F_1}{\partial V_1^2} + \tfrac{1}{2} F_1 V_1 \cdot \frac{\partial H_1}{\partial V_1}\right] = \tilde{J}(F, FH), \tag{37}$$

with the operator $\tilde{J}$ defined in (28).

Depending on the sign of the real part of the leading eigenvalue γ, the function h will either decay (when $\mathrm{Re}(\gamma) < 0$) or grow (when $\mathrm{Re}(\gamma) > 0$) with time. In the latter case the hydrodynamic solution is unstable and does not possess physical significance.

The sign of the real part of γ may be determined in the case of slightly inelastic collisions. Then, in the leading-order approximation (with respect to $\epsilon = 1 - e \ll 1$) analyses of (37) reduce to the following eigenvalue problem, posed for the linearized collision operator $\tilde{J}(F^{(0)}, F^{(0)}H)|_{e=1}$ governing the system of smooth elastic spheres:

$$\lambda F_1^{(0)} H_1 = \tilde{J}(F^{(0)}, F^{(0)}H)|_{e=1}, \tag{38}$$

where $\lambda = -K(F,F)\gamma$.

Equation (38) possesses five eigensolutions $H(V_1) = \{1, V_1, V_1^2\}$, each corresponding to $\lambda = 0$. All other values of λ are negative (see e.g. see Cercignani 1975). This result combined with (34) and the above expression for λ, yields for any $\lambda < 0$

$$\gamma = -\lambda / K(F,F) \approx \lambda / 3.86\epsilon < 0. \tag{39}$$

Passing in formulae (39), (36) (with $e_0(t)$ given by (25), (26)) to the limit $\epsilon \to 0$, one obtains that h decays exponentially with time, which reproduces the known classical exponential approach of any state to the equilibrium for a gas of smooth elastic spheres.

It may be expected that the above considerations are applicable to more general circumstances (extending beyond the case $\epsilon \ll 1$). This assertion is supported by the extensive and successful use of the hydrodynamic equations in situations for which the existence of hydrodynamic solutions was not proven. Additional support for the possibility of employing the hydrodynamic equations is provided by a relatively weak effect of inelasticity on the form of the singlet distribution function.

Physically, instability of the hydrodynamic state of the system of smooth inelastic spheres is explained by two competitive processes: (i) energy relaxation, which is kinetic energy exchange between different degrees of freedom, and (ii) dissipation, caused by the energy losses resulting from the particles' collisions. For particles close to absolutely elastic spheres the energy relaxation rate, γ_r, exceeds the dissipation rate, γ_d. With increasing inelasticity γ_r decreases and γ_d increases and for absolutely inelastic particles ($e = 0$) $\gamma_r < \gamma_d$. It is clear, that there exists a certain value of e, say e_m, such that for $e < e_m$ the relaxation process will not terminate and the evolution of the granular system will not be solely determined by the mean granular kinetic energy (temperature) and, hence, will not be hydrodynamic.

Particle roughness generally increased the number of degrees of freedom participating in the energy exchange process. Therefore, this property causes retardation of the relaxation process on the one hand, and accelerates energy dissipation on the other. Hence, in a system of rough inelastic spheres e_m should decrease with increasing roughness, which accords with the results of §3.5 [see (54*a*) below].

3.5. *Partition of granular fluctuation kinetic energy*

Partition of kinetic energy in gaseous systems composed of complex molecules (possessing many degrees of freedom) is usually analysed using models of rough and smooth loaded spheres, spherical cylinders and ellipsoids (see e.g. Theodosopulu & Dahler, 1974). These models predict equal partition of particle kinetic energy between their translational and rotational degrees of freedom. In spite of the difference between the non-conservative interactions of macroscopic granules and the conservative interactions of gaseous molecules, the assumption of equipartition of kinetic energy is used in the statistical models developed for fluidized (Goldshtik & Kozlov 1973; Nigmatullin 1978) and magnetofluidized (Buevich *et al.* 1985) beds.

Partition of particle fluctuation kinetic energy in granular systems has been examined on an *ad hoc* basis in the particular case of a steady Couette flow of disks (Jenkins & Richman 1985*b*) and spheres (Lun & Savage 1987; Lun 1991). It was shown that kinetic energy partition in such systems depends only upon particle roughness and does not depend on the inelasticity of their collisions. We will investigate the partition of fluctuation energy on the basis of a rigorous analysis of the kinetic equation (15) for the collisional model characterized by constant e, β. In this case $\epsilon_i = 0$ and (20) adopts the form

$$-K(F,F)\left(3F_1+\frac{\partial F_1}{\partial V_1^2}V_1^2+\frac{\partial F_1}{\partial \Omega_1^2}\Omega_1^2\right) = \tilde{J}(F,F), \tag{40}$$

where
$$\tilde{J}(F,F) = \int d^6\mu_2\, d^2k(\boldsymbol{k}\cdot\boldsymbol{v}_{21})\,\theta(\boldsymbol{k}\cdot\boldsymbol{v}_{21})\,[(e\beta)^{-2}F_1''F_2''-F_1F_2] \tag{41}$$

is the dimensionless collisional integral and operator $K(F,F)$ is calculated by introducing (A 6) in (19) and subsequent integration with respect to vector $\boldsymbol{k}$:

$$K(F,F) = \frac{\pi}{4}\left[\eta_2(\eta_2-1)+\frac{\eta_2^2}{k}-\frac{1-e^2}{4}\right]\int d^6\mu_1\, d^6\mu_2\, V_{21}^3 F_1 F_2 + \frac{\pi}{3}\frac{\eta_2}{k}\left[\eta_2\left(\frac{k+1}{k}\right)-1\right]\int d^6\mu_1\, d^6\mu_2\, V_{21}(\Omega_1^2+\Omega_2^2)F_1F_2, \tag{42}$$

with η_1, η_2 given by (A 5).

Equations (40)–(42) together with normalization conditions (21) were first derived by Goldshtein *et al.* (1990). We go beyond the latter study and analyse these equations for the case of arbitrary rough spheres. In this case the solution of (40) (similarly to the procedure employed for solution of (27)) will be sought in the form of expansion in the series of the Sonine polynomials, i.e.

$$F(V_1^2, \Omega_1^2, e, \beta) = \frac{1}{\pi^3(\alpha_t\,\alpha_r)^{3/2}} \exp(-\tilde{V}_1^2 - \tilde{\Omega}_1^2) \sum_{i,j} a_{ij}\, S_{1/2}^{(i)}(\tilde{V}_1^2)\, S_{1/2}^{(j)}(\tilde{\Omega}_1^2), \tag{43}$$

$$\tilde{V}_1^2 = V_1^2/\alpha_t, \quad \tilde{\Omega}_1^2 = \Omega_1^2/\alpha_r$$

where $a_{ij}, \alpha_t, \alpha_r$ are dimensionless coefficients, dependent upon e, β, and $k = 4I/m\sigma^2$.

Parameters α_t, α_r are subjected to the following normalization condition:

$$\alpha_t + \alpha_r = \tfrac{4}{3}, \tag{44}$$

assuring the proper behaviour of the energy partition in the limiting cases $e = |\beta| = 1$ (Chapman & Cowling 1970).

Combination of (43) and normalization conditions (21), (44) yields

$$a_{00} = 1, \quad a_{10} = a_{01} = 0. \tag{45}$$

After satisfying the normalization condition (44), one obtains the first term in (43) in the form of the Maxwell–Boltzmann distribution with parameters α_t, α_r, governing granular kinetic translational, T_t and rotational, T_r temperatures:

$$T_t = \tfrac{1}{2}e_0\,\alpha_t, \quad T_r = \tfrac{1}{2}e_0\,\alpha_r, \quad T_t + T_r = \tfrac{2}{3}e_0. \tag{46}$$

It follows from (46), that $\frac{1}{2}\alpha_t$ and $\frac{1}{2}\alpha_r$ may be interpreted as the respective translational and rotational constant-pressure specific heats.

Definitions of translational and rotational kinetic energy temperatures, given by (17), (43), (46) are natural generalizations of comparable definitions employed in conservative systems to the case of dissipative granular systems. Physically it is clear that these generalizations are justifiable when the energy relaxation rate due to kinetic energy transfer between the translational and rotational modes is larger than the energy decay rate due to kinetic energy dissipation. In the opposite case the energy exchange between the two modes is effectively 'frozen', occurring, as it does, much slower than the processes of evolution of hydrodynamic quantities. Mathematically this may be manifested by the fact that the relative contribution of the first term (Maxwell–Boltzmann distribution) of (43) is of the same order as the contributions of the subsequent 'higher-order' terms. Alternatively, in the frozen energy exchange case the translational and rotational constant pressure specific heats α_t, α_r may exhibit physically unplausible functional dependences of e and β (e.g. α_t, α_r may obtain negative values). However, bearing in mind fast convergence of (48) in the particular case of smooth spheres, as will not examine the higher-order terms in (43) and will limit ourselves by the Maxwell–Boltzmann approximation of function F also in a more general case of rough spheres.

Equation (40) possesses several useful properties, which will prove helpful in finding its solutions. Upon multiplication of both sides of (40) by the summational invariants of a perfectly elastic rough sphere system, i.e. 1, V, $\frac{1}{2}(V_1^2 + \Omega_1^2)$ and integration over the velocity space, this equation turns into identity. Therefore, the solution of this equation satisfying momental relationships up to the second order must include only the first

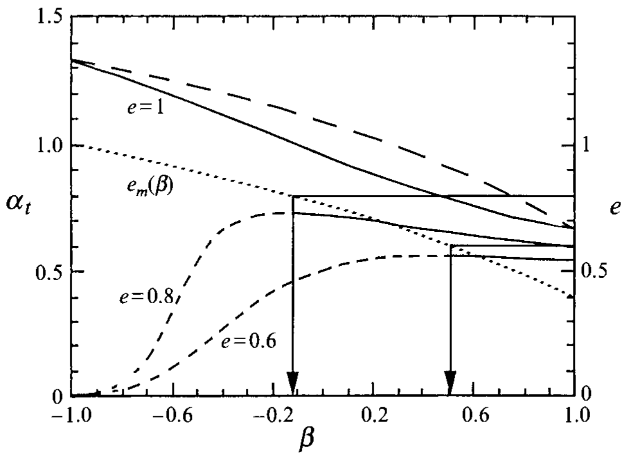

FIGURE 1. The dependence of the partition coefficient, α_t, upon the particle roughness β and inelasticity e for $k = 0.4$ (uniform spheres). The upper broken line is from Lun & Savage (1987), $0 < e \leqslant 1$, (equation (48 *c*)). The dotted line $e = e_m(\beta)$ describes the range of validity of the hydrodynamic solution. The applicability range of each curve (marked by straight lines) is to the right of the corresponding vertical arrow.

term in expansion (43). Calculating any one of the second-order moments V_1^2 or Ω_1^2, one obtains the following relationship between parameters α_t, α_r:

$$a\alpha_t \alpha_r = \tfrac{2}{3}b(\alpha_t - \alpha_r), \tag{47a}$$

where

$$a = (1-\beta^2)\frac{1-k}{1+k} - 1 + e^2, \quad b = 2k\left(\frac{1+\beta}{1+k}\right)^2. \tag{47b, c}$$

Equations (44), (47 *a*) may be replaced by a single quadratic equation, governing either of parameters α_t, α_r. The solution for this equation is subjected to the following physical condition : for absolutely elastic spheres ($e = 1$) the kinetic translational temperature exceeds the rotational temperature ($T_t > T_r$). From this requirement one obtains

$$\alpha_t = \frac{2}{3}\left(1 + \frac{a}{b + (a^2+b^2)^{1/2}}\right), \quad \alpha_r = \frac{2}{3}\left(1 - \frac{a}{b + (a^2+b^2)^{1/2}}\right). \tag{48a, b}$$

Expression (48 *a*) for α_t may be compared with the formula

$$\alpha_t = \frac{4}{3}\left(\frac{9-5\beta}{11-3\beta}\right), \tag{48c}$$

derived by Lun & Savage (1987) and Lun (1991) for $k = 0.4$. In contrast with the latter studies, in the present model the kinetic temperatures were found to depend not only upon the roughness of granules but also upon their inelasticity.

Figure 1 depicts the dependence of α_t upon the roughness β for several values of the inelasticity parameter e. For $e = 1$ the difference between the present results and (48 *c*) does not exceed 13%. However, for decreasing inelasticity the partition coefficient exhibits a strong dependence on e, especially for almost smooth particles (β close to -1). In particular, for $e = 0.6$, and intermediate values of the roughness coefficients ($\beta = -0.1$) α_t predicted by (48 *c*) is overestimated by as much as 55%. This error is indeed very large since it is comparable to the overall effect of β on α_t within the whole range of the roughness parameter.

Coefficients α_t, α_r are found to be monotonic functions of inelasticity (see figure 1). However, these parameters exhibit nonmonotonic β-dependences. Parameter α_t reaches maximum on the curve

$$e = e_m(\beta) = \left(\frac{2k}{1+k} - \beta\frac{1-k}{1+k}\right)^{1/2}. \tag{49a}$$

The non-monotonic character of the β-dependence of α_t may be explained by the following qualitative considerations. Observe from (A 6) that the translational kinetic energy loss $(\Delta E_t)_{diss}$ in a particle collision is independent of β. However, the comparable rotational kinetic energy loss $(\Delta E_r)_{diss} \propto (1-\beta^2)$, and hence increases with β increasing from -1 to 0. Assume that at $t = 0$ the kinetic energies of the rotational and the translational modes are of the same order, $E_r \sim E_t$. Then for almost perfectly smooth ($B \gtrsim -1$) but inelastic ($e = e^* < 1$) particles E_t decreases with time much more rapidly than E_r. Since for such particles the average kinetic energy exchanged between the rotational and translational modes, $(\Delta E)_{exch}$, is very small, the total kinetic energy rapidly redistributes in favour of the rotational modes. With β increasing from $\beta \gtrsim -1$ to $\beta \sim 0$ (and $e = e^* = \text{const}$), one obtains that $(\Delta E_r)_{diss} \propto (1-\beta^2)$ increases, while $(\Delta E_t)_{diss} \propto (1-e^2)$ (see (A 6)) and, hence, remains constant. Therefore with increasing β, E_r (or, equivalently, α_r) decreases and E_t (or α_t) increases, in accordance with the trend observed in figure 1. With β further increasing from 0, $(\Delta E_r)_{diss}$ decreases, which tends to increase E_r. However, $(\Delta E)_{exch}$ increases with increasing β, which leads to a more equal partition of the kinetic energy. These two competitive tendencies result in a weak β-dependence of α_r, which is clearly seen in figure 1.

Since the above considerations have a qualitative nature, the maximum of α_t (minimum of α_r) occurs at a certain roughness $\beta = \beta_m(e)$, which differs from zero. We will estimate $\beta_m(e)$ directly from the collisional model. Assume that α_t reaches maximum when $(\Delta E_r)_{diss} = (\Delta E_t)_{diss}$, or (see (A 6))

$$\frac{1-e^2}{4} g_{21k}^2 \approx \left(\frac{1-\beta^2}{1+k}\right)\frac{k}{4} g_{21r}^2.$$

Assuming further, than on average $g_{21k}^2 \approx g_{21r}^2$, one obtains

$$\beta = \beta_m(e) = \pm\left(1-(1-e^2)\frac{1+k}{k}\right)^{1/2}.$$

Taking the negative root of the above expression and expanding it in a series for $(1-e^2) \ll 1$, one obtains

$$\beta_m(e) = \frac{1-k}{2k} - e^2\frac{1+k}{2k}.$$

Comparing the above expression with (54a), rewritten in the form

$$\beta_m(e) = \frac{2k}{1-k} - e^2\frac{1+k}{1-k}, \tag{49b}$$

one can conclude that both formulae predict shifting of the maximum value β_m towards $\beta = -1$, with increasing e.

In the limiting case of absolutely smooth spheres, $\beta = -1$ there is no kinetic energy exchange between the translational and rotational degrees of freedom. This means, that the behaviour of the system composed of such particles is independent of the value of E_r, which will be explicitly and implicitly absent in the hydrodynamic equations.

This means that the rotational temperature, i.e. the kinetic energy participating in the energy exchange process, $T_r = 0$, which implies $\alpha_r = 0$. On the other hand, our model yields $\alpha_r = \frac{4}{3}$. The inability of the present model to reproduce the above limiting results constitutes its defect, which should be resolved in the framework of a more elaborated model of particle collisions (Jenkins 1992).

The effect of increasing particle inelasticity $(1-e)$ is to increase the kinetic energy, $(\Delta E_t)_{diss}$, lost in each collision by the translational degrees of freedom. As a result, the granular temperature is redistributed in favour of the rotational modes (see figure 1).

The properties of the partition parameters α_t, α_r, discussed above, need experimental verification as well as further theoretical investigation based on more elaborated collisional models (e.g. the model described by (1), which includes the dependences of coefficients e, β upon the particles' relative impact velocity $\boldsymbol{g}_{21}$). Such a study lies beyond the scope of this work; however, it may be performed by the calculational method developed above.

3.6. *Range of existence of the hydrodynamic solution*

The process of temporal evolution of a system of rough inelastic spheres to the hydrodynamic state is characterized by a timescale t_1, in which the system achieves the 'equilibrium' energy partition, given by (48*a*, *b*). It is obvious that this time is inversely proportional to the kinetic energy, $(\Delta E)_{exch}$, exchanged between the different modes during each collision. The latter energy is larger for rougher particles and smaller for smoother ones. In particular, for $\beta \rightarrow -1$ t_1 is large and may significantly exceed the characteristic time t_2 of the kinetic energy decay due to the dissipation process. Mathematically this is manifested by a decrease of T_t (and, hence, α_t) with decreasing β. That is, for those parts of the curves $\alpha_t(e, \beta)$, where α_t decreases with decreasing $\beta(\beta < \beta_m)$, plotted as broken lines in figure 1, no hydrodynamic solution exists. In further treatment we will consider only the range of parameters $\beta > \beta_m$, or $e > e_m(\beta)$, with $e_m(\beta)$ given by (49*a*), where the present hydrodynamic solution exists and possesses physical significance. The dependence (49*a*) is shown in figure 1 and may be used to graphically determine the range of applicability of any of $\alpha_t(e, \beta)$ curves.

It must be noted that existence of the hydrodynamic solution for granular systems was discussed (Jenkins & Richman 1985*b*; Lun 1991) in terms of small inelasticity, $1-e \ll 1$. These qualitative estimates are quantitatively scrutinized here by our determination of the range of existence of the hydrodynamic solution in the (e, β)-plane.

For the collisional model employed here the proposed hydrodynamic solution (17) depends only on the total energy of random motion. This assumption automatically implies that the translational and rotational kinetic fluctuation energies (e_{0t}, e_{0r}) and temperatures (T_t, T_r) depend on time according to the formula

$$T_t, T_r, e_{0t}, e_{0r} \sim \frac{1}{[1-K(F,F)\,t/(2t_0)]^2}, \tag{50}$$

with $K(F, F)$ obtained in accordance with (42) and the Maxwell–Boltzmann approximation of function F (see (43)):

$$-K(F,F) = (\tfrac{1}{2}\pi)^{1/2}\alpha_t^{3/2}\left[1-e^2+\left(\frac{1-\beta^2}{1+k}\right)\left(k+\frac{\alpha_r}{\alpha_t}\right)\right]. \tag{51}$$

This quantity may be used for evaluating the accuracy of the hydrodynamic solution $f_1^{(0)}$ which may be performed similarly to the case of smooth spheres, considered in

§3.4. It is further shown that $K(F, F)$ characterizing the decay rate of the kinetic energy of the particle random motion also governs the intensity of the zero-order volumetric sink term, appearing in the Euler-like energy equation (see (79 c, d)).

The approximate solution for the singlet distribution function, given by the first term of (48), satisfies all momental relationships (or their combinations) up to the second order. Therefore, consideration of other energy balance laws (separate for each mode, etc.) will not provide any new insight. Bearing in mind the above observation and assuming sufficiently fast convergence of expansion (43), we will further limit ourselves by the accepted Maxwell–Boltzmann approximation of the hydrodynamic solution for $f^{(0)}$.

Now it is possible to calculate the dependence of the isotropic, collisional part, $\boldsymbol{P}^{(c)}$ of the pressure tensor upon the density and the energy e_0 in the spatially homogeneous state. This may be done by substituting the Maxwell–Boltzmann approximation (17), (43) for $f_1^{(0)}$ with coefficients a, b, α_t, α_r given by (47 b, c), (48 a, b) into (11 a) and neglecting the spatial non-locality (expressed by the terms including $\sigma \boldsymbol{k}$) in all integrand functions. The above function $\boldsymbol{P}^{(c)}(e_0, n)$ will be evaluated in the following section by the Chapman–Enskog method, which will also systematically give the higher-order contributions to $\boldsymbol{P}^{(c)}$ with respect to gradients of the hydrodynamic properties.

4. Derivation of the Euler-like equations for a moving granular medium by the Chapman–Enskog method

4.1. *Solution scheme*

Consider the kineti stage of evolution of a system composed of rough inelastic spheres. In this stage the system state is determined by the singlet distribution function $f_1 = f(\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{\omega}_1, t)$, which obeys the equation

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}_1 \cdot \frac{\partial f_1}{\partial \boldsymbol{x}} = J(f, f, \boldsymbol{x}), \tag{52}$$

obtained from (2) by invoking the molecular chaos hypothesis (5). Using (3) for the collisional integral combined with (4), evaluated for the collisional model chosen here as $\Lambda = e^{-1}\beta^{-2}$ (Goldshtein *et al.* 1990), one can rewrite the right-hand side (52) in the form

$$J(f, f, \boldsymbol{x}) = \int \mathrm{d}^6\tau_2\, \mathrm{d}^2k S(\boldsymbol{k}\cdot\boldsymbol{v}_{21}) [(e\beta)^{-2} g(\boldsymbol{x} + \tfrac{1}{2}\sigma\boldsymbol{k}) f_1''(\boldsymbol{x}) f_2''(\boldsymbol{x} + \sigma\boldsymbol{k}) - g(\boldsymbol{x} - \tfrac{1}{2}\sigma\boldsymbol{k}) f_1(\boldsymbol{x}) f_2(\boldsymbol{x} - \sigma\boldsymbol{k})]. \tag{53}$$

We assume that after a certain period the system reaches the hydrodynamic evolution stage, where its state is weakly inhomogeneous and completely described by the first five moments $\zeta_\alpha(\boldsymbol{x}, t)$, appearing in the right-hand side of (14), of the singlet distribution function. This means that the gradients of the hydrodynamic quantities obey the following estimates:

$$\partial^n \zeta_\alpha(\boldsymbol{x}, t)/\partial \boldsymbol{x}^n \sim \phi^n, \quad \phi \ll 1, \tag{54}$$

where $\phi \sim L^{-1}$, with L being a characteristic sale of inhomogeneities existing within the system.

We will apply the Chapman–Enskog method to systematically construct the solution for (52), (53). Specifically, we will derive the dependences of the hydrodynamic fluxes upon the corresponding properties in the zero- and first-order approximations with respect to the spatial gradients of $\zeta_\alpha(x, t)$.

Classical applications of the Chapman–Enskog method to gaseous systems involve expansions in terms of a small parameter $Kn = l/L$, where l is the particle (molecule) mean free path. An appropriate non-dimensional form of (52) involves the characteristic timescale L/a, where a is the speed of sound in gaseous systems. In the dense granular media considered here, a strongly depends upon the density n and hence may not serve to define the above-mentioned dimensionless parameters. Therefore, we will apply the Chapman–Enskog method to (52) and (53) in their dimensional forms, noting that it leads to results identical to those obtained after non-dimensionalizing of these equations.

In accordance with the basic idea of the Chapman–Enskog method, the singlet distribution function is represented as a powers series in ϕ:

$$f_1 = f(\boldsymbol{v}_1, \boldsymbol{\omega}_1, \zeta_\alpha(\boldsymbol{x}, t)) = f_1^{(0)} + f_1^{(1)} + O(\phi^2), \qquad (55)$$

where $f^{(n)} = O(\phi^n)$, $n = 0, 1, \ldots,$ and all functions $\zeta_\alpha(\boldsymbol{x}, t) = \{n(\boldsymbol{x}, t), \boldsymbol{u}(\boldsymbol{x}, t), e_0(\boldsymbol{x}, t)\} = O(1)$.

The above implies the following normalization conditions imposed on $f^{(n)}$:

$$\int \mathrm{d}^6\boldsymbol{\tau}_1 f_1^{(0)} = n, \quad \int \mathrm{d}^6\boldsymbol{\tau}_1\, \boldsymbol{v}_1 f_1^{(0)} = n\boldsymbol{u}, \quad \int \mathrm{d}^6\boldsymbol{\tau}_1\, \boldsymbol{\omega}_1 f_1^{(0)} = 0, \qquad (56\,a\text{–}c)$$

$$\int \mathrm{d}^6\boldsymbol{\tau}_1 f^{(0)}[\tfrac{1}{2}|\boldsymbol{v}_1 - \boldsymbol{u}|^2 m + \tfrac{1}{2} I\omega^2] = n e_0, \qquad (56\,d)$$

$$\int \mathrm{d}^6\boldsymbol{\tau}_1 f_1^{(1)} = \int \mathrm{d}^6\boldsymbol{\tau}_1 f_1^{(1)} \boldsymbol{v}_1 = \int \mathrm{d}^6\boldsymbol{\tau}_1 f_1^{(1)} \boldsymbol{\omega}_1 = \int \mathrm{d}^6\boldsymbol{\tau}_1 f_1^{(1)} \left[\tfrac{1}{2}|\boldsymbol{v}_1 - \boldsymbol{u}|^2 + \frac{I\omega^2}{2m}\right] = 0. \qquad (57)$$

Strictly speaking, hydrodynamic solution (55) with normalization conditions (56) and (57) corresponds to flows which are free of externally imposed torques. For obtaining a more general theory, accounting for the latter quantities, in (55)–(57) one should replace $\boldsymbol{\omega}$ by $\boldsymbol{\omega} - \boldsymbol{\omega}_0$, where $\boldsymbol{\omega}_0$ is the mean particle spin. One may assume that $\boldsymbol{\omega}_0$ is equal to the fluid angular velocity, i.e. $\boldsymbol{\omega}_0 = \frac{1}{2}\operatorname{rot}\boldsymbol{u}$, which is accurate up to small values of the second order with respect to the gradients of hydrodynamic quantities (Chapman & Cowling 1970; McCoy *et al.* 1966). Accounting for the particle spin does not change the form of the Euler-like hydrodynamic equation for flowing granular media (Goldshtein *et al.* 1990).

It follows from the assumptions (14), (54), that $f_1^{(1)}$ appearing in (55) may be represented as a linear combination of the gradients of the hydrodynamic properties ζ_α, the exact form of which combination will be determined below. Smallness of the gradients of $\zeta_\alpha(\boldsymbol{x}, t)$,$f[\boldsymbol{v}_1, \boldsymbol{\omega}_1, \zeta_\alpha(\boldsymbol{x}, t)]$ can be used for expanding the expressions for the pressure tensor P_{ij}, the heat flux $\boldsymbol{j}$, the sink term I, and the collisional integral J in series with respect to small parameter ϕ:

$$P_{ij} = P_{ij}^{(0)}(f^{(0)}) + P_{ij}^{(1)}(f^{(0)}) + P_{ij}^{(1)}(f^{(1)}) + \ldots, \qquad (58\,a)$$

$$\boldsymbol{j} = \boldsymbol{j}^{(0)}(f^{(0)}) + \boldsymbol{j}^{(1)}(f^{(0)}) + \boldsymbol{j}^{(1)}(f^{(1)}) + \ldots, \qquad (58\,b)$$

$$I = I^{(0)}(f^{(0)}) + I^{(1)}(f^{(0)}) + I^{(1)}(f^{(1)}) + \ldots, \qquad (58\,c)$$

$$J(f, f, \boldsymbol{x}) = J^{(0)}(f^{(0)}, f^{(0)}) + J^{(1)}(f^{(0)}, f^{(0)}) + J^{(1)}(f^{(0)}, f^{(1)}) + \ldots. \qquad (58\,d)$$

Here superscript (0), (1), etc. denote the orders with respect to ϕ. In particular, $P_{ij}^{(1)}(f^{(0)})$ is the first-order contribution to the stress tensor, arising from the zero-order term in expansion (55) for the singlet distribution function f. The second terms in the right-hand sides of (58) appear due to *non-local* effects arising from the first-order corrections to f (see e.g. (53)), and the comparable third terms are due to the *local* effects, arising from the first-order corrections to f.

Introducing expansions (58*a–c*) into (13), and omitting terms of $O(\phi^n)$, $n > 1$, one obtains the hydrodynamic equations of the Euler type

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \boldsymbol{x}}\cdot(n\boldsymbol{u}), \quad \frac{\partial u_j}{\partial t} = -u_i\frac{\partial u_j}{\partial x_i} - \frac{1}{\rho}\frac{\partial P_{ij}^{(0)}(f^{(0)})}{\partial x_i}, \tag{59a, b}$$

$$\frac{\partial e_0}{\partial t} = -\boldsymbol{u}\cdot\frac{\partial e_0}{\partial \boldsymbol{x}} - \frac{1}{n}\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{j}^{(0)}(f^{(0)}) - \frac{P_{ij}^{(0)}(f^{(0)})}{n}\frac{\partial u_i}{\partial x_j} + \frac{1}{n}[I^{(0)}(f^{(0)}) + I^{(1)}(f^{(0)}) + I^{(1)}(f^{(1)})]. \tag{59c}$$

Unlike the classical Chapman–Enskog solution, in the present case of rough inelastic granular gas, the zero-order (Euler-like) equations are determined not only by the zero-order function $f^{(0)}$ but also by $f^{(1)}$, which was overlooked in previous studies (Jenkins & Richman 1985*b*; Lun & Savage 1987; Lun 1991). In contrast to the usual Euler equation, the kinetic energy transport equation (59*c*) includes a sink term, the structure of which is determined by $f^{(0)}$ and $f^{(1)}$.

Equations governing $f^{(0)}$, $f^{(1)}$. Equations (59*a–c*) may be used to expand the time derivative of the singlet distribution function f_1 in the series

$$\frac{\partial f_1}{\partial t} = \frac{\partial f_1^{(0)}}{\partial \zeta_\alpha}\frac{\partial \zeta_\alpha}{\partial t} + \frac{\partial f_1^{(1)}}{\partial t} + O(\phi^2). \tag{60a}$$

The spatial gradient of $f_1^{(0)}$ may be expressed via the comparable gradients of hydrodynamic properties ζ_α

$$\frac{\partial f_1^{(0)}}{\partial \boldsymbol{x}} = \frac{\partial f_1^{(0)}}{\partial \zeta_\alpha}\frac{\partial \zeta_\alpha}{\partial \boldsymbol{x}}. \tag{60b}$$

Introducing (60*a, b*) together with (58*a–d*) into (52), one obtains the equations governing $f^{(0)}$ and $f^{(1)}$:

$$\frac{I^{(0)}(f^{(0)})}{n}\frac{\partial f_1^{(0)}}{\partial e_0} = J(f^{(0)}, f^{(0)}), \tag{61}$$

$$\frac{\partial f_1^{(0)}}{\partial n}\left[\boldsymbol{C}_1\cdot\frac{\partial n}{\partial \boldsymbol{x}} - n\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{u}\right] + \frac{\partial f_1^{(0)}}{\partial u_j}\left[\boldsymbol{C}_1\cdot\frac{\partial u_j}{\partial \boldsymbol{x}} - \frac{1}{\rho}\frac{\partial P_{ij}^{(0)}(f^{(0)})}{\partial x_i}\right] - J^{(1)}(f^{(0)}, f^{(0)})$$
$$+ \frac{\partial f_1^{(0)}}{\partial e_0}\left[\boldsymbol{C}_1\cdot\frac{\partial e_0}{\partial \boldsymbol{x}} - \frac{1}{n}\frac{\partial}{\partial \boldsymbol{x}}\cdot\boldsymbol{j}^{(0)}(f^{(0)}) - \frac{P_{ij}^{(0)}(f^{(0)})}{n}\frac{\partial u_i}{\partial x_j} + \frac{I^{(1)}(f^{(0)})}{n}\right]$$
$$= J^{(1)}(f^{(0)}, f^{(1)}) - \frac{\partial f_1^{(0)}}{\partial e_0}\frac{I^{(1)}(f^{(1)})}{n} - \frac{\partial f_1^{(1)}}{\partial t}. \tag{62}$$

Solution of (62) requires knowledge of $f_1^{(0)}$, explicitly appearing therein. Therefore, we will first solve (61) for $f_1^{(0)}$, which will be done in the following subsection.

4.2. *Contributions of the zero-order function to the hydrodynamic equations*

According to its definition, $f^{(0)}$ is independent of spatial gradients of the hydrodynamic properties. Therefore, solution of (62) subject to the normalization condition (56) should possess a form identical to that describing the spatially homogeneous state. As was shown in §3.5, the latter solution may be represented by the Maxwell–Boltzmann distribution function

$$f_1^{(0)} = \frac{n(mI)^{3/2}}{e_0^3\pi^3}\frac{1}{(\alpha_t\alpha_r)^{3/2}}\exp(-\tilde{V}_1^2-\tilde{\Omega}_1^2), \tag{63}$$

where $\tilde{V}_1 = (\boldsymbol{v}_1-\boldsymbol{u})[m/(e_0\alpha_t)]^{1/2}$, $\tilde{\boldsymbol{\Omega}}_1 = \omega_1[I/(e_0\alpha_r)]^{1/2}$. The zero-order sink term, appearing in the right-hand side of (59*c*) was calculated in §3.5 (cf. (18), (51)):

$$I^{(0)}(f^{(0)}) = K(F,F)\,\sigma^2 n\rho g(n)(e_0/m)^{3/2}. \tag{64}$$

Now it is possible to calculate other contributions of $f^{(0)}$ to the Euler-like hydrodynamic equations. Similar to the classical Chapman–Enskog solution (Chapman & Cowling 1970), the zero-order solution (63) yields zero heat diffusion flux and an isotropic stress tensor:

$$P_{ij}^{(0)} = \delta_{ij}P, \quad \boldsymbol{j}^{(0)} = 0. \tag{65}$$

The hydrostatic pressure P in a granular medium consists of a purely 'kinetic' part, P_k and a collisional transfer part, P_c:

$$P = P_k + P_c, \quad P_k = nT_t, \quad P_c = \tfrac{1}{3}\pi(1+e)\,\sigma^3 n^2 g(n)\,T_t. \tag{66a–c}$$

It is seen from (66*c*) that the effect of inelasticity of particle collisions is to decrease the collisional transfer pressure part (with respect to the case of elastic collisions) by the factor $\frac{1}{2}(1+e)$. In addition, both inelasticity and roughness affect P_k and P_c via the translational temperature T_t.

Introduce solution (63) into (12) and use (58*c*) to obtain the first-order contribution of $f^{(0)}$ to the kinetic energy dissipation

$$I^{(1)}(f^{(0)}) = N(F)\,P_c\,\nabla\cdot\boldsymbol{u}, \tag{67}$$

with

$$N(F) = \tfrac{3}{2}(1-e) + \left(\frac{1-\beta^2}{1+k}\right)\left(\frac{k+\alpha_r/\alpha_t}{1+e}\right). \tag{68}$$

This term is associated with energy losses arising from compression of the medium. Goldshtein *et al.* (1990) obtained (67) without specifying the β-dependence of $N(F)$ in (68).

The first-order contribution of $f^{(0)}$ to the collisional integral (58*d*) is obtained by introducing (63) into (53) and employing (58*d*):

$$J^{1)}(f^{(0)},f^{(0)}) = f_1^{(0)}\left(\frac{e_0}{m}\right)^{1/2}\frac{P_c}{P_k}\left\{\boldsymbol{A}_1(F)\cdot\frac{\partial\ln P_c}{\partial\boldsymbol{x}} - [\boldsymbol{B}_1(F)+7\boldsymbol{A}_1(F)]\cdot\frac{\partial\ln e_0}{\partial\boldsymbol{x}} - (m/e_0)^{1/2}[H_{u1}\nabla\cdot\boldsymbol{u} + \boldsymbol{H}_{u1}:\nabla^0\boldsymbol{u}]\right\}, \tag{69}$$

where the dimensionless vectors $\boldsymbol{A}_1$, $\boldsymbol{B}_1$, the traceless tensor $\boldsymbol{H}_{u1}$ and the scalar H_{u1} are respectively defined by

$$\boldsymbol{A}_1(F) \equiv \frac{1}{2}\int \mathrm{d}^6\tau_2\,\mathrm{d}^2k\,\tilde{S}(\boldsymbol{k}\cdot\boldsymbol{V}_{21})\,\boldsymbol{k}\left(F_2 + \frac{F_1'' F_2''}{e^2\beta^2 F_1}\right), \tag{70a}$$

$$\boldsymbol{B}_1(F) \equiv -\int \mathrm{d}^6\tau_2\,\mathrm{d}^2k\,\tilde{S}(\boldsymbol{k}\cdot\boldsymbol{V}_{21})\,\boldsymbol{k}\left[(\tilde{V}_2^2+\tilde{\Omega}_2^2)F_2 + \frac{F_1'' F_2''}{e^2\beta^2 F_1}(\tilde{V}_2''^2+\tilde{\Omega}_2''^2)\right], \tag{70b}$$

$$\begin{Bmatrix}\boldsymbol{H}_{u1}\\ H_{u1}\end{Bmatrix} \equiv \int \mathrm{d}^6\tau_2\,\mathrm{d}^2k\,\tilde{S}(\boldsymbol{k}\cdot\boldsymbol{V}_{21})\left[\frac{F_1'' F_2''}{e^2\beta^2\alpha_t F_1}\begin{Bmatrix}2\boldsymbol{k}^0\boldsymbol{V}_2''\\ \frac{2}{3}\boldsymbol{k}\cdot\boldsymbol{V}_2''\end{Bmatrix} + \frac{F_2}{\alpha_t}\begin{Bmatrix}2\boldsymbol{k}^0\boldsymbol{V}_2\\ \frac{2}{3}\boldsymbol{k}\cdot\boldsymbol{V}_2\end{Bmatrix}\right], \tag{70c, d}$$

with $\tilde{S}(\boldsymbol{k}\cdot\boldsymbol{V}_{21}) = \theta(\boldsymbol{k}\cdot\boldsymbol{V}_{21})(\boldsymbol{k}\cdot\boldsymbol{V}_{21})/b$ and $b=\frac{1}{3}\pi(1+e)$. In the above, $\boldsymbol{k}^0\boldsymbol{V}_2$ is the traceless symmetric part of the dyadic $\boldsymbol{k}\boldsymbol{V}_2$.

Equations (66)–(71) describe the contribution of the zero-order function to (62), which is required in the hydrodynamic Euler-like equations. The solution of (62) will be obtained in the following subsection.

4.3. *The Euler-like hydrodynamic equations*

Introduce $J^{(1)}(f^{(0)},f^{(0)})$ given by (69), (70*a–d*), and (65), (63) for $P_{ij}^{(0)}$, $\boldsymbol{j}^{(0)}$, $f_1^{(0)}$, respectively, into (62) to obtain the following equation for $f^{(1)}$:

$$f_1^{(0)}\,\nabla\cdot\boldsymbol{u}\left\{\frac{2}{\alpha_t}L(\mu_1)-1+\tfrac{2}{3}\tilde{V}_1^2+\frac{P_c}{P_k}\left[H_{u1}+\frac{2}{\alpha_t}[1-N(F)]\,L(\mu_1)\right]\right\}$$
$$+f_1^{(0)}\,\nabla^0\boldsymbol{u}:\left[\frac{P_c}{P_k}\boldsymbol{H}_{u1}+2\tilde{\boldsymbol{V}}_1{}^0\tilde{\boldsymbol{V}}_1\right]-f_1^{(0)}\left(\frac{e_0}{m}\right)^{1/2}\frac{1}{n}\frac{\partial\ln P_c}{\partial\boldsymbol{x}}\cdot[\boldsymbol{V}_1+\boldsymbol{A}_1(F)]$$
$$+f_1^{(0)}\left(\frac{e_0}{m}\right)^{1/2}\frac{\partial\ln e_0}{\partial\boldsymbol{x}}\left\{\boldsymbol{V}_1[1-L(\mu_1)]+\frac{P_c}{P_k}[\boldsymbol{B}_1(F)+7\boldsymbol{A}_1(F)]\right\}$$
$$=\frac{f_1^{(0)}}{ne_0}L(\mu_1)\,I^{(1)}(f^{(1)})-\frac{\partial f_1^{(1)}}{\partial t}+J^{(1)}(f^{(0)},f^{(1)}),\quad (71)$$

where $L(\mu_1)=(3-\tilde{V}_1^2-\tilde{\Omega}_1^2)$. Comparing the left-hand side of this equation with the left-hand side of the corresponding equation obtained for absolutely elastic rough spheres (McCoy *et al.* 1966), one can see that the effects of e and β manifest themselves in the appearance of an additional term, proportional to $\partial\ln P_c/\partial\boldsymbol{x}$. These collisional properties also affect the terms in the left-hand side of (71), including gradients of $\boldsymbol{u}$ and $\ln e_0$. Goldshtein *et al.* (1990) formally obtained in (71) an additional term proportional to ∇n. This, however, does not contradict the treatment presented here, since the difference between $f^{(0)}$ and the Maxwell–Boltzmann function (63) is negligibly small (see also the discussion in §3.5).

The right-hand side of (71) consists of three linear operators applied to $f^{(1)}$. The last one is the linearized collisional operator

$$J^{(1)}(f^{(0)},f^{(1)}) = g(n)\int d^6\tau_2\, d^2k S(\boldsymbol{k}\cdot\boldsymbol{v}_{21})\,[(f_1^{(0)\prime\prime}f_2^{(1)\prime\prime}+f_1^{(1)\prime\prime}f_2^{(0)\prime\prime})\,(e\beta)^{-2}$$
$$-(f_1^{(0)}f_2^{(1)}+f_1^{(1)}f_2^{(0)})].\quad (72)$$

The first two terms in the right-hand side of (71) have no analogues in the models characterized by conservative collisions. The first of them results from the contribution of the source term $I^{(1)}(f^{(1)})$. The second time-derivative term in the right-hand side of (71) may be shown to be determined by $I^{(0)}(f^{(0)})$, that is, by particle collisional energy losses. Towards this goal construct the solution for $f_1^{(1)}$ as a most general linear combination of the spatial gradients of the hydrodynamic properties ζ_α:

$$f_1^{(1)} = -\frac{f_1^{(0)}}{n}\left[\boldsymbol{A}_1^{(1)}\cdot\frac{\partial\ln n}{\partial\boldsymbol{x}}+\boldsymbol{B}_1^{(1)}\cdot\frac{\partial\ln P_c}{\partial\boldsymbol{x}}+\boldsymbol{C}_1^{(1)}\cdot\frac{\partial\ln e_0}{\partial\boldsymbol{x}}\right.$$
$$\left.+\left(\frac{m}{e_0}\right)^{1/2}(D_1^{(1)}\,\nabla\cdot\boldsymbol{u}+\boldsymbol{E}_1^{(1)}:\nabla^0\boldsymbol{u})\right],\quad (73)$$

where $\boldsymbol{A}_1^{(1)}$, $\boldsymbol{B}_1^{(1)}$, $\boldsymbol{C}_1^{(1)}$, $D_1^{(1)}$, $\boldsymbol{E}_1^{(1)}$ are functions of the dimensionless velocities V_1, Ω_1. In addition, these coefficient functions depend parametrically upon ζ_α (but not on $\partial\zeta_\alpha/\partial \boldsymbol{x}$), and upon the mechanical and geometric properties of granules and the interactions between them. Vector coefficients $\boldsymbol{A}^{(1)}, \boldsymbol{B}^{(1)}, \boldsymbol{C}^{(1)}$ contribute to the constitutive equation for the kinetic energy flux. This equation will include, in addition to the 'Fourier' term proportional to $\partial \ln e_0/\partial \boldsymbol{x}$, two other terms, respectively proportional to $\partial \ln n/\partial \boldsymbol{x}$ and $\partial \ln P_c/\partial \boldsymbol{x}$. Since in the two classical cases $e = |\beta| = 1$ these terms are absent (Chapman & Cowling 1970) their nature is governed by the particle collisional kinetic energy loss. Finally, the tensor coefficient $\boldsymbol{E}^{(1)}$ contributes to the viscosity, and the scalar $D^{(1)}$ determines the bulk viscosity of the granular medium.

Further it follows from (59) and definition (60*a*) that the time-derivative term appearing in the right-hand side of (71) possesses the following form:

$$\begin{aligned}\frac{\partial f_1^{(1)}}{\partial t} = -\frac{I^{(0)}(f^{(0)})}{n^2}\bigg\{&\frac{\partial}{\partial e_0}[f_1^{(0)}\boldsymbol{A}_1^{(1)}]\cdot\frac{\partial \ln n}{\partial \boldsymbol{x}} + \frac{\partial}{\partial e_0}[f_1^{(0)}\boldsymbol{B}_1^{(1)}]\cdot\frac{\partial \ln P_c}{\partial \boldsymbol{x}}\\ &+\frac{\partial}{\partial e_0}[\boldsymbol{C}_1^{(1)}f_1^{(0)}]\cdot\frac{\partial \ln e_0}{\partial \boldsymbol{x}} + \frac{\partial}{\partial e_0}\left[D_1^{(1)}f_1^{(0)}\left(\frac{m}{e_0}\right)^{1/2}\right]\nabla\cdot\boldsymbol{u}\\ &+\frac{\partial}{\partial e_0}\left[\boldsymbol{E}_1^{(1)}f_1^{(0)}\left(\frac{m}{e_0}\right)^{1/2}\right]:\nabla^0\boldsymbol{u}\bigg\} - \frac{f_1^{(0)}}{n}\boldsymbol{C}_1^{(1)}\cdot\frac{\partial}{\partial \boldsymbol{x}}\left[\frac{I^{(0)}(f^{(0)})}{n}\right],\end{aligned} \quad (74)$$

which explicitly includes the collisional sink term $I^{(0)}(f^{(0)})$.

Using (73) and (59*c*), the term $I^{(1)}(f^{(1)})$ may be shown to be proportional to the divergence of the hydrodynamic velocity $\boldsymbol{u}$:

$$I^{(1)}(f^{(1)}) = -\nabla\cdot\boldsymbol{u}\left(\frac{e_0}{m}\right)^{1/2}\frac{g(n)}{2}\int \mathrm{d}^6\tau_1\,\mathrm{d}^6\tau_2\,\mathrm{d}^2k S(\boldsymbol{k}\cdot\boldsymbol{v}_{21})[D_1^{(1)}f_2^{(0)} + D_2^{(1)}f_1^{(0)}]\Delta E, \quad (75)$$

with $D_i^{(1)} = D^{(1)}(\boldsymbol{V}_1, \boldsymbol{\Omega}_i)$, $i = 1,2$. This result is explained (Goldshtein *et al.* 1990) by the oddness of the functions $\boldsymbol{A}_1^{(1)}, \boldsymbol{B}_1^{(1)}, \boldsymbol{C}_1^{(1)}$ (with respect to $\boldsymbol{V}_1$), appearing in the right-hand side of (73).

According to (69), (71), the last term in the right-hand side of (74) for $\partial f_1^{(1)}/\partial t$ may be represented as a linear combination of $\partial \ln n/\partial \boldsymbol{x}$, $\partial \ln P_c/\partial \boldsymbol{x}$ and $\partial \ln e_0/\partial \boldsymbol{x}$. Using the linearity property of the operator $J(f^{(0)}, f^{(1)})$, defined by (72), and expression (75) for $I^{(1)}(f^{(1)})$ combined with (74), one can obtain from (71), a set of non-homogeneous integro-differential equations for the unknown functions $\boldsymbol{B}^{(1)}$, $\boldsymbol{C}^{(1)}D^{(1)}$, $\boldsymbol{E}^{(1)}$, and a homogeneous integro-differential equation governing $\boldsymbol{A}^{(1)}$. Normalization conditions for these equations may be obtained by introducing solution (73) into (57). Approximate solutions for these equations may be obtained by traditional methods, e.g. by expansion in terms of Sonine polynomials. Equations for $\boldsymbol{B}^{(1)}, \boldsymbol{C}^{(1)}, \boldsymbol{A}^{(1)}, \boldsymbol{E}^{(1)}$ are necessary for constructing the higher-order hydrodynamic equations of the Navier–Stokes type.

Calculation of sink term $I^{(1)}(f^{(1)})$. We will limit ourselves by considering the equation for $D^{(1)}$, since this function governs the sink term $I^{(1)}(f^{(1)})$, required for closure of the hydrodynamic Euler-like equations (59*a–c*). The solution for $D_i^{(1)} = D^{(1)}(\boldsymbol{V}_i, \boldsymbol{\Omega}_i)$ possesses the form (see Appendix B)

$$D_i^{(1)} = \frac{1}{\sigma^2 g(n)}\left[D_{ki} + \frac{P_c}{P_k}D_{ci}\right], \quad i = 1,2, \quad (76a)$$

$$D_{ki} = a_{kt}S_{1/2}^{(1)}(\tilde{V}_i^2) + a_{kr}S_{1/2}^{(1)}(\tilde{\Omega}_i^2), \quad D_{ci} = a_{ct}S_{1/2}^{(1)}(\tilde{V}_i^2) + a_{cr}S_{1/2}^{(1)}(\tilde{\Omega}_i^2). \quad (76b)$$

Coefficients $a_{kt}, a_{ct}, a_{kr}, a_{cr}$ are calculated in Appendix B (see (B 12), (B 13*a*, *b*)). In the range of parameters β and e where the hydrodynamic solution exists, coefficients a_{kt}, a_{ct} are monotonic, slowly varying functions of β and e (see figure 7).

Introduce expressions (76*a*, *b*) for $D_i^{(1)}$ into (75) to obtain the following formula for source term $I^{(1)}(f^{(1)})$:

$$I^{(1)}(f^{(1)}) = (C_1 P_k + \lambda_2 P_c)\nabla\cdot\boldsymbol{u}, \tag{77}$$

where

$$C_1 = a_{kt}\lambda, \quad \lambda_2 = a_{ct}\lambda, \tag{78a, b}$$

$$\lambda = -(\tfrac{1}{2}\pi\alpha_t)^{1/2}\left[3(1-e^2)+\left(\frac{1-\beta^2}{1+k}\right)\left(3k-2+\frac{\alpha_r}{\alpha_t}\right)\right]. \tag{78c}$$

Introduce expressions (65*a*, *b*) for the pressure tensor and the diffusional part of the heat flux, and expressions (64), (67), (77) for sink terms into (59*a–c*) to obtain the Euler-like hydrodynamic equations of a medium consisting of inelastic rough spheres:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u}) = 0, \quad \frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\frac{1}{\rho}\nabla P = 0, \quad \frac{\partial e_0}{\partial t}+\boldsymbol{u}\cdot\nabla e_0+\frac{P}{n}\nabla\cdot\boldsymbol{u} = \frac{I(E)}{n}, \tag{79a–c}$$

$$I(E) = K(F,F)\,\sigma^2 g(n)\,\rho(e_0/m)^{3/2}+(C_1 P_k + C_2 P_c)\nabla\cdot\boldsymbol{u}, \tag{79d}$$

where

$$C_2 = \lambda_2 + N(F), \tag{79e}$$

and $K(F,F)$, C_1, λ_2, $N(F)$ are respectively given by (51), (78*a*, *b*) and (68).

These equations constitute the main goal of the present analysis. They can be used for mathematical description of a wavy motion of granular materials and, in particular, for modelling of particle motion in vibrofluidized beds (Goldshtein *et al.* 1993).

For the model of rough inelastic spheres the expression for the zero-order sink term was obtained by Lun (1991). In the following section his result will be compared with the one obtained in the present study. Other sink terms for the collisional model employed here were not considered in the previous studies.

5. Volumetric kinetic energy sink terms

5.1. *The sink term characterizing energy decay in a homogeneous state*

This term is represented by the first member in the right-hand side of (79*c*), with $K(F,F)$ given by (51). It describes the random motion energy losses within a granular medium flowing 'incompressibly'.

Figure 2 depicts the dependence of the normalized value of the above sink term, represented by coefficient $K(F,F)$, upon the roughness β for several values of inelasticity e. These curves are drawn in the domain of existence of the hydrodynamic solution. For $e > 0.6$ (approximately) each of the curves possesses a well-established extremum point, whih falls into the above domain. This non-monotonic behaviour of $K(F,F)$ is explained by the β-dependence of the energy losses resulting from particle collisions. When particles are almost smooth (β close to -1) or almost absolutely rough (β close to 1) the kinetic energy losses due to interparticle friction are small. Accordingly, for these particles all energy losses are associated with the inelasticity of collisions. However, at intermediate values of the roughness (β close to 0) the frictional part of the collisional energy losses reaches a maximum, resulting also in a maximum of the kinetic energy decay rate $K(F,F)$.

Lun (1991) obtained the first member in the right-hand side of formula (79*d*) with the coefficient $K(F,F)$ in the form identical to (51), although with wrong e- and β-

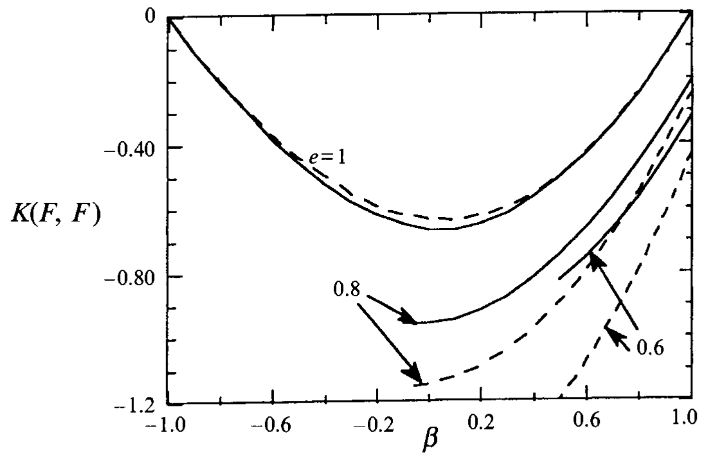

FIGURE 2. The dependence of the normalized value of the sink term in the spatially uniform state of a granular medium (see (64)) upon the particle inelasticity e and roughness β for $k = 0.4$ (uniform spheres). Broken lines represent the solution with the energy partition function obtained by Lun (1991).

dependences of the coefficients α_r, α_t. Accordingly, the values of the normalized sink term $K(F, F)$ calculated by the theory of Lun (1991) differ significantly from the present results. A comparison presented in figure 2 shows that within the range of existence of the hydrodynamic solution the results of Lun (broken lines) overestimate the kinetic energy losses as much as by 50%.

5.2. *Sink terms characterizing energy losses arising from compression of granular medium*

(*a*) The term $I^{(1)}(f^{(0)}) = N(F)\, P_c \nabla \cdot \boldsymbol{u}$. The appearance of the velocity divergence in the above sink term suggests that it arises from the compression work performed by the pressure forces, i.e. by the collisional pressure P_c. Although this interpretation seems to be supported by comparing (67) with the term $P\nabla \cdot \boldsymbol{u}$ appearing in the classical Euler equation (cf. also (79*c*)), the nature of the sink term (67) is different. The effect of this term is seen to diminish e_0 during gas compression, which, according to the above interpretation, would be equivalent to a negative compression work, which is impossible. To understand the origination of sink term (67), observe that $\nabla \cdot \boldsymbol{u}$ is proportional to the rate of increase of gas density and, therefore, to the rate of increase of particle collisions. Each of these collisions is accompanied by kinetic energy losses. Therefore, $N(F)$ in (67) represents normalized kinetic energy loss (gain) due to increase (decrease) of particles' collision rate, arising from gas compression (expansion).

Figure 3 depicts the dependence of the coefficient $N(F)$, governing the intensity of sink term (67), with irrelevant parts of the calculated curves (where the hydrodynamic solution does not exist) shown as broken lines. One can see that for elastic spheres ($e = 1$) the kinetic energy losses predicted by (67), are relatively low (N is below 0.3). With increasing inelasticity (decreasing e) the intensity of this sink term increases owing to increasing collisional losses. All curves possess maxima, which are located approximately at the same roughness coefficient $\beta \approx 0.2$. These maxima do not occur for the most dissipative particles, i.e. at $\beta = 0$, as one would expect, since the coefficients α_t, α_r both depend on β (see (67)).

Upon passing to the limit $\beta = 1$ in (68) for $N(F)$, one obtains the expression $N(F) =$

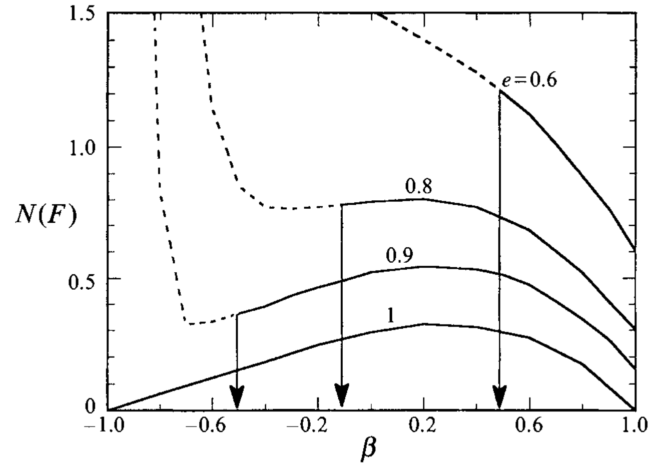

FIGURE 3. The dependence of the normalized value of the sink term arising from compression of a granular medium (see (67)) upon the particle inelasticity e and roughness β for $k = 0.4$ (uniform spheres). For the applicability range of each curve see caption for figure 1.

$1.5(1-e)$ obtained by Jenkins & Richman (1985*a*) for the particular case of smooth ($\beta = -1$) particles. In the present model the latter limit cannot be achieved (except for the case $e = 1$; see discussion of this matter in §3.5).

One can see that (68) predicts quite a strong β-dependence of the sink term $I^{(1)}(f^{(0)})$ with respect to the form of this term obtained by Jenkins & Richman (1985*a*). For inelastic particles characterized by $\beta \approx 0.2$, disregarding this dependence can result in a several-fold error in calculation of $I^{(1)}(f^{(0)})$. Moreover, for elastic particles $N(F)$ obtained by Jenkins & Richman (1985*a*) vanishes, whereas (68) predicts non-zero energy losses, except for the limiting cases $\beta = \pm 1$.

(*b*) The term $I^{(1)}(f^{(1)}) = (C_1 P_k + \lambda_2 P_c)\nabla\cdot\boldsymbol{u}$. This sink term, appearing within the last member of the right-hand side of (79*d*), represents additional energy losses arising from compression of granular media. This term constitutes a new result, since it was not obtained in any previous studies of the collisional transport of granular materials.

The term $I^{(1)}(f^{(1)})$ consists of two parts respectively proportional to the kinetic and the collisional granular pressures. These pressures depend in a different manner upon the granular gas density, which determines the relative magnitudes of the two components of this sink term. However, coefficients C_1, λ_2, are both proportional to λ. The latter quantity, thus, determines an additional kinetic energy dissipation rate, associated with gas compression, and will be analysed below.

To analyse this term we will rewrite (77) via the relaxational part $\eta_b^{(1)}$ of the bulk viscosity, given by (B 15*c*), thereby obtaining

$$I^{(1)}(f^{(1)}) = -\lambda\nabla\cdot\boldsymbol{u}\frac{3\eta_b^{(1)}}{\pi(1+e)\,\sigma}\left(\frac{2T_t}{m\alpha_t}\right)^{1/2}. \tag{80}$$

One can see from (80) that $I^{(1)}(f^{(1)})$ is proportional to the product $\eta_b^{(1)}\nabla\cdot\boldsymbol{u}$ and, hence, is related to the kinetic energy relaxation phenomenon, governing also the value of the bulk viscosity term $\eta_b^{(1)}$ (see Appendix B). Coefficient λ given by (79*c*) and appearing in (80) determines the influence of the energy relaxation upon the total energy balance in the system, and may be, thus, interpreted as a normalized value of the sink term $I^{(1)}(f^{(1)})$.

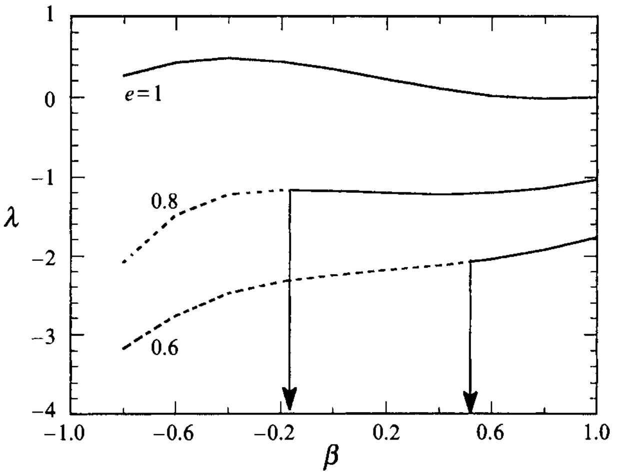

FIGURE 4. The dependence of the correction factor λ for the kinetic energy partition in the spatially uniform state of a granular medium, arising from relaxation phenomena (see (77), (78*a*, *b*)) upon the particle inelasticity e and roughness β for $k = 0.4$ (uniform spheres). For the applicability range of each curves see caption for figure 1.

Figure 4 shows the dependence of λ upon β for several values of e in the range of existence the hydrodynamic solution (i.e. the range of monotonic dependence of parameters α_t, α_r upon e, β). For $e = 1$ the effect of gas compression ($\nabla \cdot \boldsymbol{u} < 0$) is to pump additional kinetic energy into the random granular motion of the system, as seen in figure 4 ($\lambda > 0$). One can see that in spite of the collisional energy losses associated with the particle roughness, the energy of the system is increased with respect to the case $\beta = 1$. To understand this phenomenon note that the partition of the kinetic energy of particle random motion between the rotational and translational modes given by (48*a*, *b*) was obtained for the homogeneous state. Formula (80), however, describes a non-homogeneous state of granular material, where the compression work is directly transmitted into the kinetic energy of random motion, primarily assoiated with particle translational degrees of freedom. Hence, in the presence of gas compression the kinetic energy partition law, given by (48*a*, *b*), is distorted. Therefore, the dissipation rate predicted by the $K(F, F)$ term of (51) (arising from the zero-order 'equilibrium' solution) differs from the dissipation rate prevailing during gas compression. The term $I^{(1)}(f^{(1)})$, thus, constitutes a *correction* to the zero-order dissipation term given by (64).

In the case $e = 1$ all kinetic energy losses are associated with the rotational motion, since the kinetic energy of particle translational motion is not lost during collisions. During gas compression rotational degrees of freedom possess less kinetic energy (than in the spatially uniform, 'equilibrium' case), and, hence, the corresponding collisional kinetic energy losses should be *less*. As such, for $e = 1$ the correction, given by the first-order source term $I^{(1)}(f^{(1)})$ is positive, i.e. describes kinetic energy gain.

The source term $I^{(1)}(f^{(1)})$ possesses a maximum value within the interval $-1 < \beta < 1$, corresponding to maximum collisional losses associated with the rotational motion (cf. similar disussion about $K(F, F)$). For lower values of inelasticity ($e < 1$) $I^{(1)}(f^{(1)}) < 0$, i.e. predicts additional kinetic energy dissipation with respect to the 'equilibrium' dissipation rate given by (64). This is clearly attributed to the kinetic energy losses associated with the translational motion, which losses combine with those arising from particle friction.

Analyses show that for flows of dilute granular gases ($n \to 0$) the sink term $I^{(1)}(f^{(0)}) \sim n^2$ while $I^{(1)}(f^{(1)}) \sim n$, hence the former term is smaller. In dense granular gases $I^{(1)}(f^{(1)})$ is important for particles undergoing inelastic collisions, but for elastic particles it is normally dominated by $I^{(1)}(f^{(0)})$. Further analyses of the effects of both of these terms on the speed of sound in granular media are given in the following section.

6. Propagation of waves within granular materials

The goal of this section is to investigate the effects of particle collisional properties on the propagation of waves in granular gases. Savage (1988) considered acoustic waves, propagating with the speed of sound. Goldshtein *et al.* (1993) reported on the existence of waves propagating with a supersonic velocity. Mathematical modelling of these processes involve conditions for hydrodynamic properties on the front of the latter waves as well as the expression for the speed of sound in the granular gas.

We will use the standard method of Courant & Friedrichs (1948) to rewrite (79*a–c*) in characteristic forms. Towards this goal consider the following characteristic ordinary differential equations:

$$\mathrm{d}x/\mathrm{d}t = u+a, \quad \mathrm{d}x/\mathrm{d}t = u-a, \quad \mathrm{d}x/\mathrm{d}t = u. \tag{81\,a–c}$$

The solutions of these equations may be respectively represented in the functional forms

$$\xi_+(x,t) = C_+, \quad \xi_-(x,t) = C_-, \quad \xi_*(x,t) = C_*,$$

where constants C_+, C_-, C_*, define three characteristic directions in the (x, t)-plane. In (81)

$$a^2 = T_t\varphi(n)/m, \tag{82\,a}$$

with the function φ given by

$$\varphi(n) = \frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{P}{T_t}\right) + \tfrac{1}{2}\alpha_t \frac{P}{P_k}\left[(C_2 - C_1) + (1-C_2)\frac{P}{P_k}\right], \tag{82\,b}$$

and C_1, C_2 are given by (78*a*), (79*e*). Referring to the three directions C_+, C_-, C_*, one-dimensional versions of (79*a–c*) may be rewritten in the following characteristic forms:

$$\left.\frac{\mathrm{d}P}{\mathrm{d}t}\right|_{C_+} + a\rho \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{C_+} = \frac{PI^{(0)}(f^{(0)})}{ne_0}, \quad \left.\frac{\mathrm{d}P}{\mathrm{d}t}\right|_{C_-} - a\rho \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{C_-} = \frac{PI^{(0)}(f^{(0)})}{ne_0}, \tag{83\,a, b}$$

$$\left.\frac{1}{T_t}\frac{\mathrm{d}e_0}{\mathrm{d}t}\right|_{C_*} + \left[(1-C_1)\frac{P_k}{T_t} + (1-C_2)\frac{P_c}{T_t}\right]\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n}\right)\right|_{C_*} = \frac{I^{(0)}(f^{(0)})}{ne_0}, \tag{83\,c}$$

where the sink term $I^{(0)}(f^{(0)}, f^{(0)})$ is given by (64).

The quantity a, given by (82*a, b*) and appearing in (81), (83) is the speed of propagation of 'infinitesimal' disturbances and, by analogy with the speed of sound, a_g, in molecular gases, may be termed the speed of sound in granular media. In the limiting case $e = |\beta| = 1$ these two speeds are obviously identical, since in these circumstances (79*a–c*) reduce to the classical Euler equation of an inviscid gas, and (82*a, b*) yield the corresponding speed of sound in dense molecular gases (cf. Savage 1988, equation (19)):

$$a_g^2 = \left[\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{P}{T_t}\right) + \frac{\alpha_t}{2}\left(\frac{P}{P_k}\right)^2\right]\frac{T_t}{m}. \tag{84}$$

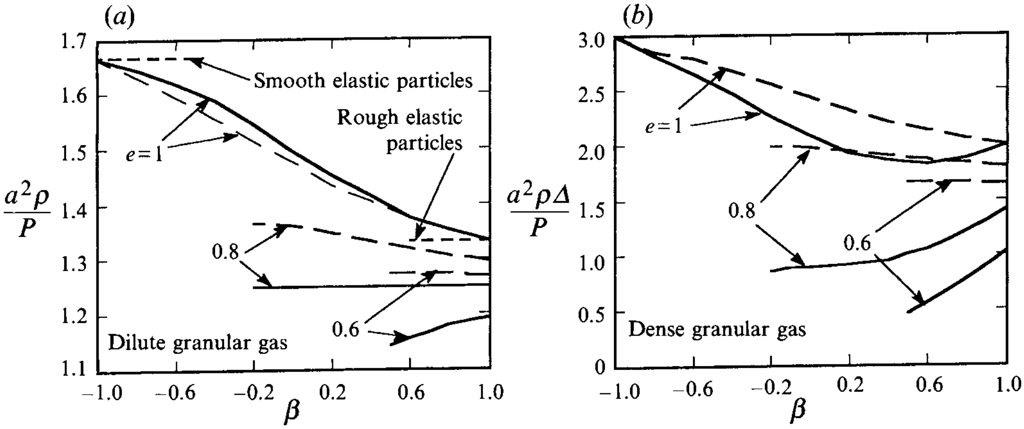

FIGURE 5. Speed of sound, a, in granular media. (a) Dilute granular gas (equation (85)); (b) granular gas with density ρ close to the maximum packing density, ρ_M, $\Delta = 1-\rho/\rho_m \ll 1$ (equation (86)). Broken lines correspond to approximate expressions for a obtained by neglecting the kinetic energy sink terms ($C_1 = 0$ in (85), $C_2 = 0$ in (86)).

Limiting case 1. *Dilute granular gas.* Analyses show that in the case of dilute gas (i.e. $P \approx P_k = nT_t$), composed of rough inelastic spheres, (82a, b) jointly with (78a) yield

$$a^2 = [1+\tfrac{1}{2}\alpha_t(1-C_1)]\frac{P}{\rho}, \tag{85}$$

where C_1 is given by (78a).

The expression in square brackets in the right-hand side of (85) is independent of density, although depends on the particle roughness and inelasticity. The effects of these parameters on the speed of sound in a dilute granular gas may be elucidated from the e- and β-dependences of α_t, a_{kt} and λ. Figure 5(a) depicts in a dimensionless form the comparable dependences of a^2. Taking in (85) the limits of perfectly elastic smooth ($\alpha_t = \frac{4}{3}, C_1 = 0$) and rough ($\alpha_t = \frac{5}{3}, C_1 = 0$) spheres, one respectively obtains $a = (4P/3\rho)^{1/2}, (5P/3\rho)^{1/2}$, with the first expression being the speed of sound in simple dilute molecular gases (Chapman & Cowling 1970).

One can see that the speed of sound a decreases with increasing inelasticity of particle collisions. This is explained by noting that the kinetic energy losses (associated with gas compression) increase with decreasing e, which energy would otherwise contribute to the process of propagation of small disturbances within the granular medium.

Setting $C_1 = 0$ in (85) one obtains $a^2 = [1+\frac{1}{2}\alpha_t]P/\rho$, which directly follows from Savage (1988, equation (84)). Results of a computation using the latter formula, shown as broken curves in figure 5(a), demonstrate that the error in the speed of sound in dilute granular gas composed of inelastic particles ($e = 0.6$), stemming from disregarding the sink term $I^{(1)}(f^{(1)})$ may amount to 12%.

Limiting case 2. *Dense granular gas.* When the gas density is close to the maximum packing density, ρ_M i.e. $\Delta = 1-\rho/\rho_M \ll 1$, one can write the equation of state for a simple gas consisting of elastic smooth spheres (Alder & Hoover, 1968) in the form $P \approx P_c = 3\rho_M T_t/(m\Delta)$. Bearing in mind that the influence of the particle inelasticity on P_c/P_k is given by the factor $\frac{1}{2}(1+e)$ (see (71)), one obtains the following approximation for the equation of state:

$$P \approx P_c = \frac{3(1+e)}{2\Delta}P_k,$$

where $P_k \approx \rho_M T_t/m$. The above, combined with (82*a, b*) yields the following approximate expression for the speed of sound:

$$a^2 = \frac{P}{\rho_M}\left[1+\frac{3\alpha_t}{4}(1+e)(1-C_2)\right]\frac{1}{\Delta}. \tag{86}$$

Expression (86) shows that for a fixed pressure P the speed of sound increases indefinitely when the gas density approaches its maximum value ($\Delta \to 0$). The ratio $a^2\rho_M \Delta/P$ is however independent of the state properties and is determined by the particle collisional parameters. This ratio is plotted in figure 5(*b*) together with its approximation obtained by disregarding the sink terms (setting $C_2 = 0$ in (86)), which directly follows from (85), obtained by Savage (1988). One can see that even in the case of elastic particles the effect of the energy dissipation on a^2 amounts to 20% (at $\beta = 0.2$). Moreover, for inelastic particles ($e = 0.6$) the error resulting from neglecting the sink terms (i.e. $C_2 = 0$) leads to overestimation of the speed of sound by more than 350%.

One can see from figure 5(*b*) that the effects of particle collisional properties on the speed of sound in dense granular materials are marked by two competitive processes: (i) redistribution of particle random-motion kinetic energy between the rotational and translational modes and (ii) kinetic energy dissipation due to gas compression. For elastic particles the first process underlies an increase of a with decreasing β, which leads to the concomitant increase of a up to the value $(3P/\rho\Delta)^{1/2}$ (at $\beta = -1$). However, kinetic energy losses rapidly increase with β decreasing from 1, which results in a diminution of a, which exhibits a minimum at about $\beta = 0.6$. For inelastic particles the energy dissipation process effectively dominates the β-dependence of the speed of sound, causing its significant diminution with β decreasing from 1.

We will, finally, derive the jump conditions for the hydrodynamic functions on the wave front. These conditions constitute an analogue of the Rankine–Hugoniot relations for the present Euler-like hydrodynamic mathematical model. For this purpose rewrite (10) in the one-dimensional form and use (66) and (79*d*) for momentum and energy fluxes and the sink term, respectively:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u) = 0, \quad \frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+P) = 0, \tag{87a, b}$$

$$\frac{\partial}{\partial t}(ne_0+\tfrac{1}{2}\rho u^2)+\frac{\partial}{\partial x}[nu(e_0+\tfrac{1}{2}mu^2)+uP] = C_0\,\sigma^2 g(n)\,\rho\left(\frac{e_0}{m}\right)^{3/2}+(C_1 P_K+C_2 P_c)\frac{\partial u}{\partial x}. \tag{87c}$$

The standard method of Courant & Friedrichs (1948) for obtaining conditions on the shock wave may be directly applied to (87*a–c*). Noting that left-hand sides of these equations are explicitly independent of particle roughness and inelasticity, and that the terms not containing gradients of the hydrodynamic functions do not affect conditions on the shock front, one obtains

$$[\rho(D-u)] = 0, \quad [\rho(D-u)^2+P] = 0, \tag{88a, b}$$

$$\left[n(D-u)\left(e_0+\frac{P}{n}+\tfrac{1}{2}m(D-u)^2\right)\right] = \lim_{h\to 0}\int_{x-h}^{x+h} \mathrm{d}\zeta (C_1 P_k+C_2 P_c)\frac{\partial u}{\partial \zeta}. \tag{89c}$$

Here, D is the shock wave velocity, $[\varphi] \equiv \varphi_{+0}-\varphi_{-0}$, with φ_{+0} and φ_{-0} being the respective values of the hydrodynamic quantities before and after the discontinuity, located in the point x. Using properties of the delta function in the right-hand side of (88), one can rewrite it in the form

$$\left[n(D-u)\left(e_0+\frac{P}{n}+\tfrac{1}{2}m(D-u)^2\right)\right] = \{C_1 P_k + C_2 P_c\}[u], \tag{90}$$

where $\{\varphi\} \equiv \frac{1}{2}(\varphi_{+0}+\varphi_{-0})$.

It follows from (90) that the particle total kinetic energy is not continuous across the shock wave front (except for the case $e = |\beta| = 1$). Depending on the sign of the expression within the curly brackets in the right-hand side of (90), this energy may either increase (due to relaxation effects) or diminish (due to dissipation of mechanical energy into thermal energy).

Equations (89) and (90) will be employed in analyses of wave propagation processes in vibrofluidized granular layers (Goldshtein *et al.* 1994*a*).

7. Discussion

The solution method developed in this paper may be compared with *ad hoc* solutions constructed by the moment methods (Jenkins & Richman 1985*a*, *b*; Lun 1991). In these methods the functional form for the singlet distribution function was adopted from the classical solutions of Grad (1949) and Chapman & Cowling (1970) independently of the particle collisional dissipative properties. This required the introduction of the restrictions of weak inelasticity ($\epsilon_e = 1-e \ll 1$) and roughness ($\epsilon_\beta = 1-|\beta| \ll 1$). Another restriction on the applicability of the hydrodynamic equations derived for fast shearing flows by the moment method is that the gradients of the hydrodynamic properties are small values of order ϵ_e, ϵ_β.

Neither of the latter restrictions is required in the solution method employed here. In particular, the second restriction imposed on the gradients of the hydrodynamic properties is not necessary in the problem of wave propagation in granular media (Goldshtein *et al.* 1994*b*).

The moment methods cannot be used for studying the influence of the kinetic energy dissipation on the structure of the hydrodynamic solution and the range of its applicability, which was done in this work. The difficulties characterizing the moment-method solution schemes increase for more complex approximations chosen for the singlet distribution function. Since in such schemes the expressions for the singlet distribution function (adopted from the classical solutions) do not contain terms proportional to $\partial \ln n/\partial x$ and $\partial \ln P_c/\partial x$, appearance of these terms in the expression for the singlet distribution function should be guessed, rather than rigorously proven. Solutions of the latter kind were discussed by Jenkins & Richman (1985*a*) and postulated by Lun *et al.* (1984), who guessed the appearance of the term $\partial \ln n/\partial x$ in the expression for the singlet distribution function.

In particular, the moment methods are shown to be incapable of deriving the right forms and obtaining numerical values of the kinetic energy dissipation terms. Namely, they do not reproduce the term $I^{(1)}(f^{(1)})$ given by (77) and fail to predict the effect of particle inelasticity on the energy partition law. The latter flaw results in overestimation of the zero-order kinetic energy losses by more than 50%. Use of incorrect expressions for the first-order kinetic energy terms, obtained by the moment methods, lead to overestimation of the speed of sound in dense granular media composed of inelastic spheres by a factor exceeding 3.

The difficulties characterizing the moment methods become still more obvious for more elaborated collisional models, e.g. when the particle collisional properties depend upon the relative impact velocity. One can see from (18) obtained for these circumstances that even in the spatially homogeneous case the singlet distibution function depends upon e_0 in a complicated manner (via coefficients $\epsilon_i = c_i(m/e_0)^{1/2}$), and, hence, the lowest-order approximation of f is no longer Maxwellian. The moment methods, however, are unable to yield these dependences (Lun & Savage 1986).

The solution method developed here is free of the difficulties associated with the choice of the right form of the singlet distribution function. On the contrary, the functional form (73) for the singlet distribution function is rigorously and unambiguously dictated by (71), derived by the Chapman–Enskog method. The present method may be easily generalized to include the dependence of particle roughness and inelasticity upon the relative collisional velocity (see §3.1) and a non-spherical particle shape. The latter geometric factor may be included along the lines delineated by Theodosopulu & Dahler (1974), who considered ellipsoidal particles. In order to perform such an investigation, an appropriate modification of the collisional integral (3) is needed.

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Appendix A. Collisions of rough inelastic spheres

Consider a configuration $(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{\omega}_1; \boldsymbol{x}_2, \boldsymbol{v}_2, \boldsymbol{\omega}_2)$ of two spheres, respectively located at positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and possessing the respective translational, $\boldsymbol{v}_1, \boldsymbol{v}_2$, and rotational, $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2$, velocities. The configuration

$$(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{\omega}_1; \boldsymbol{x}_1 - \sigma\boldsymbol{k}, \boldsymbol{v}_2, \boldsymbol{\omega}_2), \quad \boldsymbol{k}\cdot\boldsymbol{v}_{21} > 0,$$

with $\boldsymbol{v}_{21} = \boldsymbol{v}_2 - \boldsymbol{v}_1$ and the unit vector $\boldsymbol{k} = (\boldsymbol{x}_2 - \boldsymbol{x}_1)/|\boldsymbol{x}_2 - \boldsymbol{x}_1|$ directed from the centre of the second sphere to the centre of first sphere (see figure 6), describes their state just prior to the collision. At this moment the distance between their centres is equal to σ. The relative velocity of the particles at the contact point is

$$\boldsymbol{g}_{21} = (\boldsymbol{v}_2 - \tfrac{1}{2}\sigma\boldsymbol{k}\times\boldsymbol{\omega}_2) - (\boldsymbol{v}_1 + \tfrac{1}{2}\sigma\boldsymbol{k}\times\boldsymbol{\omega}_1) = g_{21k}\boldsymbol{k} + \boldsymbol{g}_{21\tau}, \qquad \text{(A 1)}$$

where $$g_{21k} = (\boldsymbol{v}_{21}\cdot\boldsymbol{k}), \quad \boldsymbol{g}_{21\tau} = \boldsymbol{v}_{21} - \boldsymbol{k}(\boldsymbol{v}_{21}\cdot\boldsymbol{k}) - \tfrac{1}{2}\sigma\boldsymbol{k}\times(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2).$$

According to the hypothesis of stereomechanic impact (Goldsmith 1960; Lun & Savage 1987), the postcollisional relative velocity $\boldsymbol{g}'_{21}$ depends on the precollisional velocity, $\boldsymbol{g}_{21}$:

$$g'_{21k} = -e g_{21k}, \quad \boldsymbol{g}'_{21\tau} = -\beta \boldsymbol{g}_{21\tau}, \qquad \text{(A 2)}$$

where $e = e(g_{21k})$ and $\beta = \beta(g_{21k}, g_{21\tau})$ are the respective coefficients of restitution and roughness. The dependences of e, β upon relative normal and tangential velocity components, $g_{21k}, g_{21\tau}$ is assumed to be known. The laws of conservation of linear and angular momenta imply that

$$\boldsymbol{v}_1 - \boldsymbol{v}'_1 = \boldsymbol{J}/m, \quad \boldsymbol{v}_2 - \boldsymbol{v}'_2 = -\boldsymbol{J}/m, \quad \boldsymbol{\omega}'_1 - \boldsymbol{\omega}_1 = \boldsymbol{\omega}'_2 - \boldsymbol{\omega}_2 = \frac{\sigma}{2I}(\boldsymbol{k}\times\boldsymbol{J}), \qquad \text{(A 3}a, b)$$

where $\boldsymbol{J}$ is the linear momentum transferred from the second particle to the first particle during the collision. It follows, thus, that the postcollisional particle state is derived by

$$\left.\begin{aligned} \boldsymbol{v}'_1 &= \boldsymbol{v}_1 + \eta_1 g_{21k}\boldsymbol{k} + \eta_2\boldsymbol{g}_{21\tau}, \quad \boldsymbol{v}'_2 = \boldsymbol{v}_2 - \eta_1 g_{21k}\boldsymbol{k} - \eta_2\boldsymbol{g}_{21\tau}, \\ \boldsymbol{\omega}'_1 &= \boldsymbol{\omega}_1 - \frac{m\sigma}{2I}\eta_2(\boldsymbol{k}\times\boldsymbol{g}_{21\tau}), \quad \boldsymbol{\omega}'_2 = \boldsymbol{\omega}_2 - \frac{m\sigma}{2I}\eta_2(\boldsymbol{k}\times\boldsymbol{g}_{21\tau}), \end{aligned}\right\} \qquad \text{(A 4)}$$

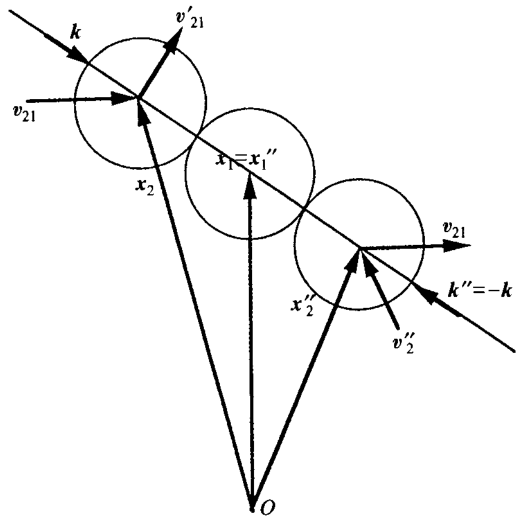

FIGURE 6. Geometry of particle collisions.

where
$$\eta_1 = \frac{1+e}{2}, \quad \eta_2 = \left(\frac{1+\beta}{1+k}\right)\frac{k}{2}. \tag{A 5}$$

One can see from (A 4), that the total momentum of the colliding particles is conserved, but their total mechanical energy E diminishes by the amount ΔE:

$$\Delta E = -m\left[\frac{1-e^2}{4} g_{21k}^2 + \left(\frac{1-\beta^2}{1+k}\right)\frac{k}{4} g_{21\tau}^2\right]. \tag{A 6}$$

Since particle collisions are irreversible, equations of impact (A 4) are not invariant with respect to interchanging the roles of the primed and non-primed variable, i.e. the sequence of states is described by the following collisional event:

$$(\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{\omega}_1; \boldsymbol{x} - \sigma\boldsymbol{k}, \boldsymbol{v}_2, \boldsymbol{\omega}_2) \xrightarrow{\boldsymbol{k}\cdot\boldsymbol{v}_{21} > 0} (\boldsymbol{x}, \boldsymbol{v}_1', \boldsymbol{\omega}_1', \boldsymbol{x} - \sigma\boldsymbol{k}, \boldsymbol{v}_2', \boldsymbol{\omega}_2'), \tag{A 7}$$

where the arrow indicates the direction of temporal evolution.

Consider a precollisional state $(\boldsymbol{x}_1'', \boldsymbol{v}_1'', \boldsymbol{\omega}_1''; \boldsymbol{x}_1'' - \sigma\boldsymbol{k}'', \boldsymbol{v}_2'', \boldsymbol{\omega}_2'')$, which results in the postcollisional state $(\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{\omega}_1; \boldsymbol{x}_1 - \sigma\boldsymbol{k}, \boldsymbol{v}_2, \boldsymbol{\omega}_2)$. It is necessary, that $\boldsymbol{x}_1'' = \boldsymbol{x}_1$, $\boldsymbol{k}'' = -\boldsymbol{k}$ (see figure 6), i.e. the centre of the second particle is located at $\boldsymbol{x}_1 + \sigma\boldsymbol{k}$ instead of at $\boldsymbol{x} - \sigma\boldsymbol{k}$. The collisional event

$$(\boldsymbol{x}_1, \boldsymbol{v}_1'', \boldsymbol{\omega}_1''; \boldsymbol{x}_1 + \sigma\boldsymbol{k}, \boldsymbol{v}_2'', \boldsymbol{\omega}_2'') \xrightarrow{(\boldsymbol{k}''\cdot\boldsymbol{v}_{21}'') > 0} (\boldsymbol{x}_1, \boldsymbol{v}_1, \boldsymbol{\omega}_1; \boldsymbol{x}_1 - \sigma\boldsymbol{k}, \boldsymbol{v}_2, \boldsymbol{\omega}_2) \tag{A 8}$$

may be described similar to the event (A 7). Using collisional hypothesis (A 2) one obtains

$$g_{21k''}'' = -e'' g_{21k''}, \quad g_{21\tau}'' = -\beta'' g_{21\tau}, \tag{A 9}$$

where $e'' = e(g_{21k''}'') = \tilde{e}(g_{21k})$, $\beta'' = \beta(g_{21k''}'', g_{21\tau}'') = \tilde{\beta}(g_{21k}, g_{21\tau})$. Employing the laws of conservation of particle momenta, one can rewrite the double-primed quantities:

$$\left.\begin{aligned} \boldsymbol{v}_1'' &= \boldsymbol{v}_1 + \frac{\tilde{\eta}_1}{\tilde{e}} g_{21k}\boldsymbol{k} + \frac{\tilde{\eta}_2}{\tilde{\beta}} \boldsymbol{g}_{21\tau}, \quad \boldsymbol{v}_2'' = \boldsymbol{v}_2 - \frac{\tilde{\eta}_1}{\tilde{e}} g_{21k}\boldsymbol{k} - \frac{\tilde{\eta}_2}{\tilde{\beta}} \boldsymbol{g}_{21\tau}, \\ \boldsymbol{\omega}_1'' &= \boldsymbol{\omega}_1 - \left(\frac{\sigma}{2I}\right)\frac{\tilde{\eta}_2}{\tilde{\beta}}(\boldsymbol{k}\times\boldsymbol{g}_{21\tau}), \quad \boldsymbol{\omega}_2'' = \boldsymbol{\omega}_2 - \left(\frac{\sigma}{2I}\right)\frac{\tilde{\eta}_2}{\tilde{\beta}}(\boldsymbol{k}\times\boldsymbol{g}_{21\tau}), \end{aligned}\right\} \tag{A 10}$$

where $\tilde{\eta}_1, \tilde{\eta}_2$ may be obtained from (A 5) by replacing e, β, by respectively $\tilde{\beta}, \tilde{e}$. In turn, dependences (A 10) may be obtained from (A 4) by replacing primed and double-primed variables, and also replacing η_1, η_2 by $\tilde{\eta}_1/\tilde{e}$, $\tilde{\eta}/\tilde{\beta}$. We will call the processes described by (A 4) and (A 10) the direct and the inverse particle collisions.

Appendix B. Calculation of function $D^{(1)}$ and bulk viscosity η_b

Here we evaluate $D_1^{(1)}$ required in solution (73), and governing the source term $I^{(1)}(f^{(1)})$ appearing in (75), as well as the bulk viscosity η_b.

Since the right-hand side of (71) is a linear operator, a particular solution for $f_1^{(1)}$ appearing in (73) can be obtained in the form

$$f_{1,p}^{(1)} = -\frac{f_1^{(0)}}{n} D_1^{(1)} \nabla \cdot \boldsymbol{u}, \tag{B 1}$$

where

$$D_1^{(1)} = \left(\frac{m}{e_0}\right)^{1/2} \frac{1}{\sigma^2 g(n)} \left(D_{k1} + \frac{P_c}{P_k} D_{c1}\right), \quad D_{k1}, D_{c1} \sim \tilde{V}_1^2, \tilde{\Omega}_1^2; \quad \tilde{V}_1^2 = \frac{V_1^2}{\alpha_t}, \tilde{\Omega}_1^2 = \frac{\Omega_1^2}{a_r}, \tag{B 2}$$

and where according to (63)

$$f_1^{(0)} = \frac{n(mI)^{3/2}}{e_0^3} F_1^{(0)}, \quad F_1^{(0)} = \frac{\exp(-\tilde{V}_1^2 - \tilde{\Omega}_1^2)}{\pi^3 (\alpha_t \alpha_r)^{3/2}}. \tag{B 3}$$

In order to determine functions D_k, D_c one must solve the following integral equations:

$$H_{u1} F_1^{(0)} + \tfrac{1}{2}\alpha_t [1 - N(F)] L(F_1^{(0)}) = Z(F^{(0)}, F^{(0)} D_c), \tag{B 4}$$

$$F_1^{(0)} (\tfrac{2}{3} \tilde{V}_1^2 - 1) + \tfrac{1}{2}\alpha_t L(F_1^{(0)}) = Z(F^{(0)}, F^{(0)} D_k), \tag{B 5}$$

where H_{u1} and $N(F)$ are given by (70d) and (68), respectively, and operators L, Z are determined for any pair functions F, G by the equations

$$L(F) \equiv 3F + V^2 \frac{\partial F}{\partial V^2} + \Omega^2 \frac{\partial F}{\partial \Omega^2}. \tag{B 6}$$

$$Z(F, G) \equiv -L(F)[K(F, G) + K(G, F)] - J(F, G) - K(F, F)[\tfrac{1}{2}G + L(G)]. \tag{B 7}$$

In turn, operators $J(F, G)$, $K(F, F)$ possess the form

$$J(F, G) \equiv \int \mathrm{d}^6\mu_2 \, \mathrm{d}^2 k (\boldsymbol{k} \cdot \boldsymbol{V}_{21}) \, \theta(\boldsymbol{k} \cdot \boldsymbol{V}_{21}) [(F_1'' G_2'' + F_2'' G_1'')(e\beta)^{-2} - F_1 G_2 - F_2 G_1], \tag{B 8}$$

where $F_i = F(V_i^2, \Omega_i^2)$, $G_i = G(V_i^2, \Omega_i^2)$, $i = 1, 2$, and

$$K(F, G) \equiv -\frac{\pi}{4}\left[\frac{1-e^2}{4} + \frac{1-\beta^2}{4}\left(\frac{k}{k+1}\right)\right] \int \mathrm{d}^{12}\mu F_1 G_2 V_{21}^3 - \frac{\pi}{12}\left(\frac{1-\beta^2}{1+k}\right) \int \mathrm{d}^{12}\mu F_1 G_2 (\Omega_1^2 + \Omega_2^2) V_{21}, \tag{B 9}$$

with $\mathrm{d}^6\mu_i \equiv \mathrm{d}^3 V_i \, \mathrm{d}^3 \Omega_i$, $(i = 1, 2)$, $\mathrm{d}^{12}\mu \equiv \mathrm{d}^6\mu_1 \, \mathrm{d}^6\mu_2$. Note, that for $G = F$ (B 9) reproduces $K(F, F)$ given by (42).

Normalization conditions (57) for $f^{(1)}$ yield subsidiary conditions for unknown functions D_k, D_c:

$$\int d\mu_1 F_1^{(0)} D_{k1} = \int d\mu_1 F_1^{(0)} D_{k1}(V_1^2+\Omega_1^2) = 0, \tag{B 10a}$$

$$\int d\mu_1 F_1^{(0)} D_{c1} = \int d\mu_1 F_1^{(0)} D_{c1}(V_1^2+\Omega_1^2) = 0. \tag{B 10b}$$

Condiff *et al.* (1965) demonstrated rapid convergence rates of the Sonine polynomials expansions of functions like D_k in the case of perfectly rough elastic spheres ($e = \beta = 1$). In particular, the lowest approximation for D_k, including a linear combination of Sonine polynomials $S_{1/2}^{(1)}(\tilde{V}_1^2)$, $S_{1/2}^{(1)}(\tilde{\Omega}_1^2)$, provides quite an accurate estimate of the bulk viscosity η_b. Similarly to the case $e = \beta = 1$, we will use the approximation

$$D_{ki} = a_{kt} S_{1/2}^{(1)}(\tilde{V}_i^2)+a_{kr} S_{1/2}^{(1)}(\tilde{\Omega}_i^2), \quad D_{ci} = a_{ct} S_{1/2}^{(1)}(\tilde{V}_i^2)+a_{cr} S_{1/2}^{(1)}(\tilde{\Omega}_i^2), \quad i = 1,2 \tag{B 11}$$

also for a more general collisional model ($0 < e \leqslant 1, -1 \leqslant \beta \leqslant 1$) employed here. One can obtain from (B 10), (B 11) conditions relating parameters a_{kt} and a_{kr}, a_{ct} and a_{cr} with α_t, α_r:

$$\alpha_t a_{kt}+\alpha_r a_{kr} = 0, \quad \alpha_t a_{ct}+\alpha_r a_{cr} = 0. \tag{B 12}$$

Two additional conditions for determination of $a_{kt}, a_{kr}, a_{ct}, a_{cr}$ may be obtained from (B 4), (B 5) and (B 11) by employing the moment method. As a result, one obtains the expressions

$$a_{kt} = \chi_k/\chi, \quad a_{ct} = \chi_c/\chi, \tag{B 13a, b}$$

with

$$\chi = \left(\frac{\pi\alpha_t}{2}\right)^{1/2}\left\{\tfrac{3}{4}(1-e^2)\,\alpha_t(3\alpha_t-\alpha_r)+\left(\frac{1-\beta^2}{1+k}\right)\tfrac{3}{4}[(3k-3)\,\alpha_t\,\alpha_r+\alpha_r^2-k\alpha_t^2]\right.$$
$$\left.+4\frac{\eta_2}{k}\left[3\eta_2\,\alpha_t+\left(1-\frac{\eta_2}{k}\right)(2\alpha_t-\alpha_r)\right]\right\}, \tag{B 13c}$$

$$\chi_c = -\tfrac{3}{4}\alpha_t\,\alpha_r[1-N(F)]+\frac{4\eta_2\,\alpha_t}{(1+e)\,k}\left[\eta_2+\left(\frac{\eta_2}{k}-1\right)\frac{\alpha_r}{\alpha_t}\right], \quad \chi_k = -\tfrac{3}{4}\alpha_t\,\alpha_r, \tag{B 13d, e}$$

where η_2, α_t, α_r and $N(F)$ are given by (A 5), (48*a*, *b*) and (68) respectively.

Now it is possible to evaluate the first-order contribution to the deviatoric part $\boldsymbol{P}_d$ of the pressure tensor $\boldsymbol{P}$. After substituting expressions for $f_1^{(0)}$ and $f_{1,p}^{(1)}$, respectively given by (63) and (B 1) into expansion (63*a*) for $\boldsymbol{P}$, one can evaluate $\boldsymbol{P}_d$ in the form

$$\boldsymbol{P}_d = (P-\eta_b\,\nabla\cdot\boldsymbol{u})\,\boldsymbol{\delta}, \tag{B 14}$$

with $\boldsymbol{\delta}$ the identity tensor.

Similar to the case of perfectly elastic rough spheres, bulk viscosity η_b may be shown to consist of two parts – $\eta_b^{(0)}$ and $\eta_b^{(1)}$:

$$\eta_b = \eta_b^{(0)}+\eta_b^{(1)}, \quad \eta_b^{(0)} = \tfrac{2}{3}\rho\sigma\left(\frac{T_t}{\pi m}\right)^{1/2}\left(\frac{P_c}{P_k}\right), \tag{B 15a, b}$$

$$\eta_b^{(1)} = -\frac{\alpha_t}{2}\frac{(me_0)^{1/2}}{\sigma^2 g(n)}\frac{P_c}{P_k}\left(a_{kt}+a_{ct}\frac{P_c}{P_k}\right), \tag{B 15c}$$

where the ratio P_c/P_k is a n-dependent function.

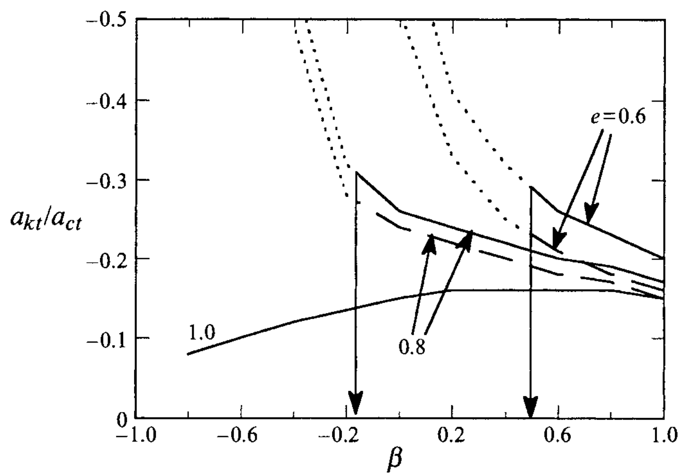

FIGURE 7. Coefficients characterizing the relaxational part of the bulk viscosity *vs.* particle roughness: a_{kt}, solid lines; a_{ct}, broken lines. For the applicability range of each curve see caption for figure 1.

In the limit $e, \beta \rightarrow 1$ expressions (48 *a, b*) for α_t, α_r and expressions (B 13) for a_{kt}, a_{ct} respectively reduce to

$$\alpha_t = \alpha_r = \alpha = \tfrac{2}{3}, \quad a_{kt} = a_{ct} = -\frac{(k+1)^2}{32k}\left(\frac{2}{\pi\alpha}\right)^{1/2}. \tag{B 16}$$

Formulae (B 15) reproduce the result of McCoy *et al.* (1966) for the bulk viscosity of perfectly elastic, perfectly rough dense sphere gas ($e = \beta = 1$).

The terms $\eta_b^{(0)}$ and $\eta_b^{(1)}$ govern different physical processes occurring in a compressing (expanding) gas. $\eta_b^{(0)}$ describes the dense gas effect, i.e. the collisional contribution of particles possessing finite dimensions into linear momentum transfer. Equation (B 15*c*) describes the contribution of the relaxation processes (i.e. exchange of kinetic energy between the rotational and the translational modes) occurring during gas expansion/compression into the pressure tensor (see discussion in §4.2).

For absolutely elastic collisions ($e = 1$) one can obtain from (B 13*c–e*) that $a_{kt} = a_{ct}$. In this case a_{kt} may be interpreted as a normalized 'relaxational' viscosity $\eta_b^{(1)}$. Parameters $a_{kt} = a_{ct}$ are plotted in figure 7 *vs.* particle roughness (curve $e = 1$). This figure also shows the values a_{kt}, a_{ct} in a more general case, where $e \neq 1$. One can see that the difference $a_{kt} - a_{ct}$ is not large for $e \leqslant 0.8$ (does not exceed 10%). Hence for 'almost elastic collisions' ($1 < e < 0.8$) each of these coefficient may still be interpreted as a normalized relaxational part of the bulk viscosity. Bearing in mind this interpretation, one can see that the bulk viscosity of slightly inelastic spheres increases with decreasing β. This may be explained by the fact that with decreasing roughness the exchange between the rotational and translational kinetic energies becomes less efficient, which results in intensification of the relaxation process, accompanied by the concomitant growth of $\eta_b^{(1)}$.

In view of the limitation of the collisional model employed here, the curves shown in figure 7 may be used in the range of e, β described by inequality $e > e_m(\beta)$.

Special consideration should be given to the case of absolutely smooth elastic spheres ($e = 1, \beta = -1$) for which both coefficients a_{kt}, a_{ct}, as well as χ_k, χ_c, χ, appearing in (B 13*c–e*) vanish, and, hence, bulk viscosity of this granular gas does not exist.

Approach to this limit is described by considering the case of slightly inelastic ($1-e \equiv \epsilon_t \ll 1$), slightly rough ($1+\beta \equiv \epsilon_r \ll 1$) spheres with $\epsilon_t, \epsilon_r \rightarrow 0$. Then, (B 13*a–e*) may be reduced to

$$a_{kt} \approx a_{ct} \approx -\frac{\alpha_r}{\chi}, \quad \chi \approx \frac{4}{3}\left(\frac{\pi}{6}\right)^{1/2}\left[6\epsilon_t + \frac{2k}{1+k}\epsilon_r\right], \quad \alpha_r \approx \frac{2k\epsilon_r^2}{3(1+k)^2(\epsilon_r k_1 - \epsilon_t)}, \quad \text{(B 17)}$$

where $k_1 = (1-k)/(1+k)$. Coefficients α_r, χ both vanish at the point $\epsilon_t = \epsilon_r = 0$; hence, the bulk viscosity in this case may not be determined (see §4.3).

The singular limit $e \rightarrow 1$, $\beta \rightarrow -1$ is characterized by an indefinite increase of the relaxation time and a simultaneous decrease of the kinetic energy transferred between the translational and rotational modes. As a result of the above competing tendencies, the value of the function $\eta_b^{(1)}$ in the point $e = 1$, $\beta = -1$ depends upon the specific path at which this limit is approached in the (e, β)-plane.

REFERENCES

ALDER, B. J. & HOOVER, W. G. 1968 Numerical statistical mechanics. In *Physics of Simple Liquids*, pp. 79–114. North-Holland.

BOGOLIUBOV, N. N. 1946 Kinetic equations. *Sov. Zh. Exp. Theor. Phys.* **16**, 691–791.

BUEVICH, YU. A., SUTKIN, S. V. & TETUKHIN, V. V. 1984 To the theory of magnetofluidized bed. *Sov. Magnit. Gidrodin.* **4**, 3–11.

CAMPBELL, C. S. 1989 The stress tensor for simple shear flows of a granular material. *J. Fluid Mech.* **203**, 449–473.

CAMPBELL, C. S. 1990 Rapid granular flows. *Ann. Rev. Fluid Mech.* **22**, 57–92.

CAMPBELL, C. S. & BRENNEN, C. E. 1985 Computer simulation of granular shear flows. *J. Fluid Mech.* **151**, 167–188.

CARNAHAN, N. F. & STARLING, K. E. 1969 Equations of state for nonattracting rigid spheres. *J. Chem. Phys.* **51**, 635–636.

CERCIGNANI, C. 1975 *Theory and Application of the Boltzmann Equation.* Scottish Academic Press.

CHAPMAN, S. & COWLING, T. G. 1970 *The Mathematical Theory of Non-Uniform Gases*, 3rd edn. Cambridge University Press.

CHLENOV, V. A. & MIKHAILOV, N. D. 1972 *Vibrofluidized Beds.* Moscow: Nauka (in Russian).

CONDIFF, D. W., LU, W. K. & DAHLER, J. S. 1965 Transport properties of polyatomic fluids, a dilute gas of perfectly rough spheres. *J. Chem. Phys.* **42**, 3445–3475.

COURANT, R. & FRIEDRICHS, K. O. 1948 *Supersonic Flow and Shock Waves.* Interscience.

DRAKE, T. G. 1990 Granular flow: physical experiments and their implication for microstructural theories. *J. Fluid Mech.* **225**, 121–152.

GOLDSHTEIN, A. I., POTURAEV, V. N. & SHULYAK, I. A. 1990 Structure of the equations of hydrodynamics for a medium consisting of inelastic rough spheres. *Fluid Dyn.* **25**, 305–313.

GOLDSHTEIN, A., SHAPIRO, M., MOLDAVSKY, L. & FICHMAN, M. 1993 Fluidized motion of vibrated beds of solid particles. In *Proc. REPFLOW II Symp. Reliable Flow of Particulate Solids II, Oslo, Norway, Aug. 23–25*, pp. 645–659.

GOLDSHTEIN, A., SHAPIRO, M., MOLDAVSKY, L. & FICHMAN, M. 1994*a* Mechanics of collisional motion of granular materials. Part 2. Wave propagation through a granular layer. *J. Fluid Mech.*

GOLDSHTEIN, A., SHAPIRO, M. & GUTFINGER, C. 1994*b* Mechanics of collisional motion of granular materials. Part 3. Self-similar wave motions. In preparation.

GOLDSHTIK, M. A. & KOZLOV, B. N. 1973 Elementary theory of concentrated disperse systems. *Sov. Prikl. Mekh. Tekh. Fiz.* **4**, 65–77.

GOLDSMITH, W. 1960 *Impact: The Theory and Physical Behavior of Colliding Solids.* E. Arnold.

GRAD, H. 1949 On the kinetic theory of rarified gas. *Commun. Pure Appl. Math.* **2**, 331–407.

GUTMAN, I. 1968 *Industrial Uses of Mechanical Vibrations.* London: Business Books.

HAFF, P. K. 1983 Grain flow as a fluid-mechanical phenomenon. *J. Fluid Mech.* **134**, 401–430.

HANLAY, H. G. M., McCARTHY, R. D. & COHEN, E. G. D. 1972 Analysis of the transport coefficients for simple dense fluids: application of the modified Enskog theory. *Physica* **60**, 322–356.

HOMSY, G. M., JACKSON, R. & GRACE, J. R. 1992 Report of a symposium on mechanics of fluidized beds. *J. Fluid Mech.* **236**, 447–495.

JENKINS, J. T. 1992 Boundary conditions for rapid granular flow: flat frictionless walls. *Trans. ASMEE: J. Appl. Mech.* **59**, 120–127.

JENKINS, J. T. & RICHMAN, M. W. 1985*a* Grad's 13-moment system for a dense gas of inelastic spheres. *Arch. Rat. Mech. Anal.* **87**, 355–377.

JENKINS, J. T. & RICHMAN, M. W. 1985*b* Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks. *Phys. Fluids* **28**, 3485–3494.

JENKINS, J. T. & SAVAGE, S. B. 1983 A theory for rapid flow of identical, smooth, nearly elastic, spherical particles. *J. Fluid Mech.* **130**, 187–202.

JOHNSON, K. L. 1982 One hundred years of Hertz contact. *Proc. Inst. Mech. Engrs* **196**, 363–378.

LUN, C. K. K. 1991 Kinetic theory for granular flow of dense, slightly inelastic, slightly rough spheres. *J. Fluid Mech.* **233**, 539–559.

LUN, C. K. K. & SAVAGE, S. B. 1986 The effect of an impact dependent coefficient of restitution on stresses developed by sheared granular materials. *Acta Mech.* **63**, 15–44.

LUN, C. K. K. & SAVAGE, S. B. 1987 A simple kinetic theory for granular flow of rough, inelastic, spherical particles. *Trans. ASMEE: J. Appl. Mech.* **54**, 47–53.

LUN, C. K. K., SAVAGE, S. B., JEFFERY, D. J. & CHEPURNIY, N. 1984 Kinetic theories of granular flow: inelastic particles in Couette flow and slightly inelastic particles in a general flow field. *J. Fluid Mech.* **140**, 223–256.

MAW, N., BARBER, J. R. & FAWCETT, J. N. 1981 The role of tangential compliance in oblique impact. *Trans ASME* **103**, 74–80.

McCOY, B. J., SANDLER, S. I. & DAHLER, J. S. 1966 Transport properties of polyatomic fluids. IV. The kinetic theory of a dense gas of perfectly rough spheres. *J. Chem. Phys.* **45**, 3485–3500.

NIGMATULLIN, R. I. 1978 *Fundamentals of the Mechanics of Heterogeneous Media*. Moscow: Nauka (in Russian).

RASKIN, KH. I. 1975 Application of the methods of physical kinetics to problems of vibrated granular media. *Dokl. Acad. Nauk SSSR* **220**, 54–57.

RICHMAN, M. W. 1989 The source of second moment of dilute granular flows of highly inelastic spheres. *J. Rheol.* **33**, 1293–1306.

SAVAGE, S. B. 1988 Streaming motions in a bed of vibrationally fluidized dry granular material. *J. Fluid Mech.* **194**, 457–478.

SONDERGAARD, R., CHANEY, K. & BRENNEN, C. E. 1990 Measurement of solid spheres bouncing of flat plates. *Trans. ASME* **57**, 694–699.

THEODOSOPULU, M. & DAHLER, J. S. 1974 The kinetic theory of polyatomic liquids. II. The rough sphere, rigid ellipsoid, and square-well ellipsoid models. *J. Chem. Phys.* **60**, 4048–4057.